

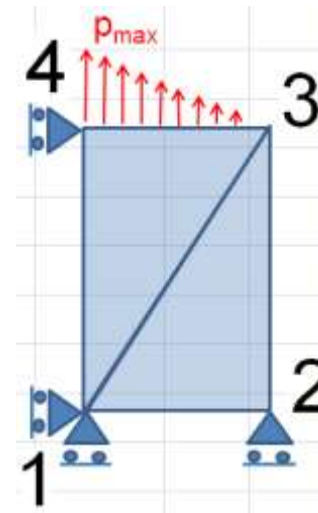
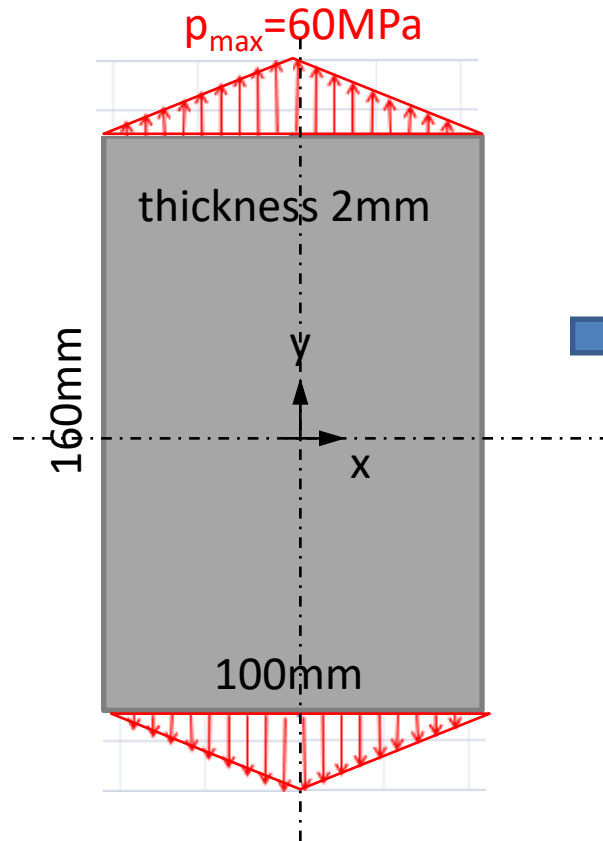


Finite element method (FEM1)

Lecture 3B. 2D Plate modeled using CST finite elements

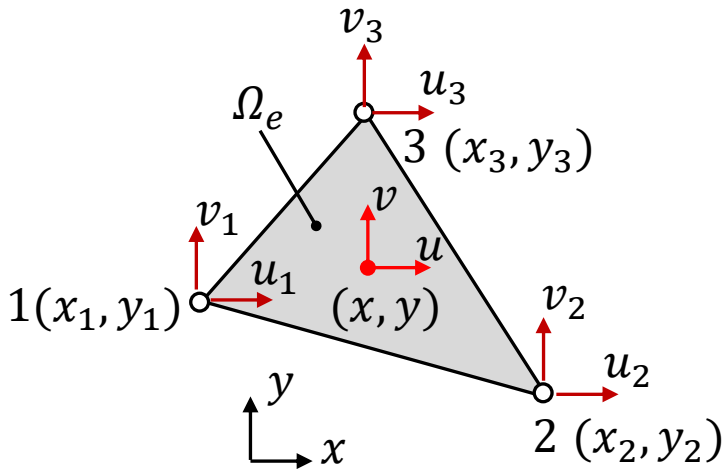
03.2025

Model of a rectangular plate



one-quarter model with
two finite elements

CST element in Plain Stress



shape functions = normalized area coordinates:

$$N_1(x, y) = \frac{A_1(x, y)}{A_e} = \frac{1}{2A_e} (a_1 + b_1x + c_1y)$$

$$N_2(x, y) = \frac{A_2(x, y)}{A_e} = \frac{1}{2A_e} (a_2 + b_2x + c_2y)$$

$$N_3(x, y) = \frac{A_3(x, y)}{A_e} = \frac{1}{2A_e} (a_3 + b_3x + c_3y)$$

where:

$$\begin{aligned} a_1 &= x_2y_3 - x_3y_2 & ; & & a_2 &= x_3y_1 - x_1y_3 & ; & & a_3 &= x_1y_2 - x_2y_1 \\ b_1 &= y_2 - y_3 & ; & & b_2 &= y_3 - y_1 & ; & & b_3 &= y_1 - y_2 \\ c_1 &= x_3 - x_2 & ; & & c_2 &= x_1 - x_3 & ; & & c_3 &= x_2 - x_1 \end{aligned}$$



$$\begin{aligned} a_i &= x_jy_k - x_ky_j \\ b_i &= y_j - y_k \\ c_i &= x_k - x_j \end{aligned}$$

Strain-displacement matrix

$$[B] = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

3×3

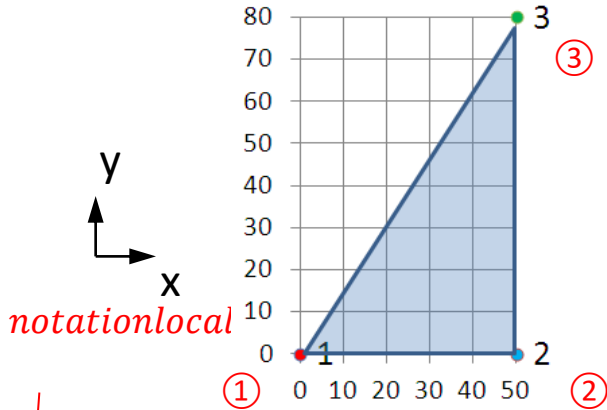
Constitutive matrix for Plain stress

local stiffness matrix of the element:

$$[k]_e = A_e t_e [B]^T [D] [B]$$

$6 \times 6 \qquad \qquad \qquad 6 \times 3 \quad 3 \times 3 \quad 3 \times 6$

Calculating the element 1 matrices



Element	1
E=	7.00E+04 MPa
ni=	0.33333333
he=	2 mm
Ae=	2000 mm ²

$$[B] = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

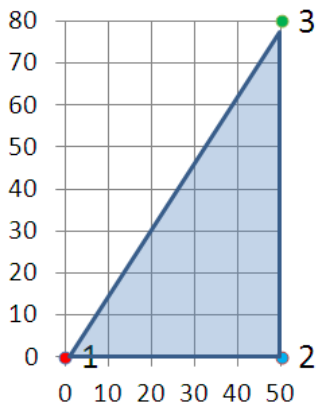
$$[D] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

node	x _i	y _i	x _j	y _j	x _k	y _k	a _i	b _i	c _i
1	0	0	50	0	50	80	4000	-80	0
2	50	0	50	80	0	0	0	80	-50
3	50	80	0	0	50	0	0	0	50

$$B_1 = \begin{bmatrix} -0.02 & 0 & 0.02 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0125 & 0 & 0.0125 \\ 0 & -0.02 & -0.0125 & 0.02 & 0.0125 & 0 \end{bmatrix}$$

$$B_1^T = \begin{bmatrix} -0.02 & 0 & 0 \\ 0 & 0 & -0.02 \\ 0.02 & 0 & -0.0125 \\ 0 & -0.0125 & 0.02 \\ 0 & 0 & 0.0125 \\ 0 & 0.0125 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 78750 & 26250 & 0 \\ 26250 & 78750 & 0 \\ 0 & 0 & 26250 \end{bmatrix}$$



Calculating the element 1 matrices

$$\mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 =$$

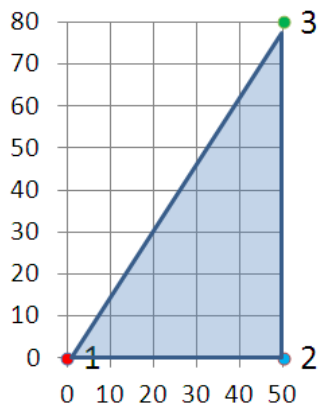
$6 \times 3 \quad 3 \times 3 \quad 3 \times 6$

	31.5	0	-31.5	6.5625	0	-6.5625
	0	10.5	6.5625	-10.5	-6.5625	0
	-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
	6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
	0	-6.5625	-4.10156	6.5625	4.101563	0
	-6.5625	0	6.5625	-12.3047	0	12.30469

Matrix multiplication example:

			D			B₁					
			78750	26250	0	-0.02	0	0.02	0	0	0
			26250	78750	0	0	0	0	-0.0125	0	0.0125
			0	0	26250	0	-0.02	-0.0125	0.02	0.0125	0
B₁^T						31.5	0	-31.5	6.5625	0	-6.5625
-0.02	0	0				0	10.5	6.5625	-10.5	-6.5625	0
0	0	-0.02				-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
0.02	0	-0.0125				6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
0	-0.0125	0.02				0	-6.5625	-4.10156	6.5625	4.101563	0
0	0	0.0125				-6.5625	0	6.5625	-12.3047	0	12.30469
0	0.0125	0									

Calculation of the stiffness matrix of element 1



$$B_1^T D B_1 =$$

	31.5	0	-31.5	6.5625	0	-6.5625
	0	10.5	6.5625	-10.5	-6.5625	0
	-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
	6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
	0	-6.5625	-4.10156	6.5625	4.101563	0
	-6.5625	0	6.5625	-12.3047	0	12.30469

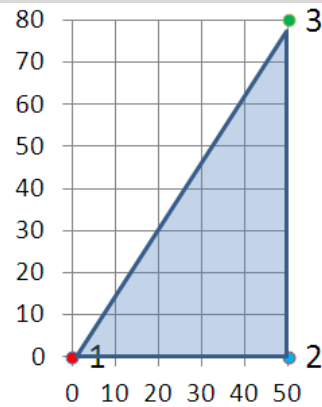
$$[k]_e = A_e t_e [B]^T [D] [B]$$

6×6 6×3 3×3 3×6

element matrix of element 1:

		u		v		u		v	
		1		1		2		2	
		3		3		3		3	
$k_1 =$	u	1	126000	0	-126000	26250	0	-26250	
	v	1	0	42000	26250	-42000	-26250	0	
	u	2	-126000	26250	142406.3	-52500	-16406.3	26250	
	v	2	26250	-42000	-52500	91218.75	26250	-49218.8	
	u	3	0	-26250	-16406.3	26250	16406.25	0	
	v	3	-26250	0	26250	-49218.8	0	49218.75	

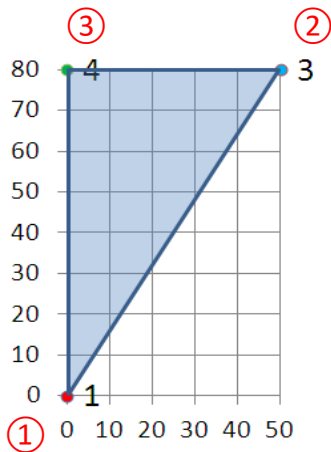
Determination of the extended stiffness matrix of element 1



extended stiffness matrix of element 1:

		u1	v1	u2	v2	u3	v3	u4	v4
k_1^*	u1	126000	0	-126000	26250	0	-26250	0	0
	v1	0	42000	26250	-42000	-26250	0	0	0
	u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
	v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
	u3	0	-26250	-16406.3	26250	16406.25	0	0	0
	v3	-26250	0	26250	-49218.8	0	49218.75	0	0
	u4	0	0	0	0	0	0	0	0
	v4	0	0	0	0	0	0	0	0

Calculating the element 2 matrices



local notation
↓

Element	2
E=	7.00E+04 MPa
ni=	0.33333333
he=	2 mm
Ae=	2000 mm ²
p _{max} =	60 MPa

$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

$$[B] = \frac{1}{2A_e} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

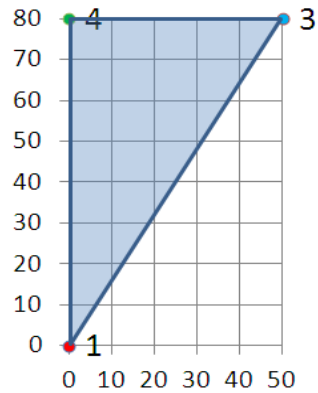
	node	x _i	y _i	x _j	y _j	x _k	y _k	a _i	b _i	c _i
①	1	0	0	50	80	0	80	4000	0	-50
②	3	50	80	0	80	0	0	0	80	0
③	4	0	80	0	0	50	80	0	-80	50

$$B_2 = \begin{bmatrix} 0 & 0 & 0.02 & 0 & -0.02 & 0 \\ 0 & -0.0125 & 0 & 0 & 0 & 0.0125 \\ -0.0125 & 0 & 0 & 0.02 & 0.0125 & -0.02 \end{bmatrix}$$

$$B_2^T = \begin{bmatrix} 0 & 0 & -0.0125 \\ 0 & -0.0125 & 0 \\ 0.02 & 0 & 0 \\ 0 & 0 & 0.02 \\ -0.02 & 0 & 0.0125 \\ 0 & 0.0125 & -0.02 \end{bmatrix}$$

$$D = \begin{bmatrix} 78750 & 26250 & 0 \\ 26250 & 78750 & 0 \\ 0 & 0 & 26250 \end{bmatrix}$$

Calculation of the stiffness matrix of element 2



$$\mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 =$$

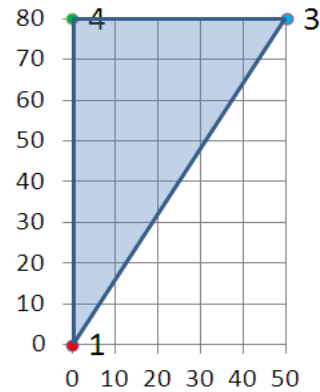
	4.1015625	0	0	-6.5625	-4.10156	6.5625
	0	12.30469	-6.5625	0	6.5625	-12.3047
	0	-6.5625	31.5	0	-31.5	6.5625
	-6.5625	0	0	10.5	6.5625	-10.5
	-4.1015625	6.5625	-31.5	6.5625	35.60156	-13.125
	6.5625	-12.3047	6.5625	-10.5	-13.125	22.80469

element matrix of element 2:

$$\mathbf{k}_2 =$$

			u	v	u	v	u	v
			1	1	3	3	4	4
u	1	16406.25	0	0	-26250	-16406.3	26250	
v	1	0	49218.75	-26250	0	26250	-49218.8	
u	3	0	-26250	126000	0	-126000	26250	
v	3	-26250	0	0	42000	26250	-42000	
u	4	-16406.25	26250	-126000	26250	142406.3	-52500	
v	4	26250	-49218.75	26250	-42000	-52500	91218.75	

Determination of the extended element stiffness matrix 2



extended stiffness matrix of element 2:

		u1	v1	u2	v2	u3	v3	u4	v4
k_2^*	u1	16406.25	0	0	0	0	-26250	-16406.3	26250
	v1	0	49218.75	0	0	-26250	0	26250	-49218.75
	u2	0	0	0	0	0	0	0	0
	v2	0	0	0	0	0	0	0	0
	u3	0	-26250	0	0	126000	0	-126000	26250
	v3	-26250	0	0	0	0	42000	26250	-42000
	u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
	v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

Determination of the global stiffness matrix

extended stiffness matrix of element 1:

		u1	v1	u2	v2	u3	v3	u4	v4
k_1^*	u1	126000	0	-126000	26250	0	-26250	0	0
	v1	0	42000	26250	-42000	-26250	0	0	0
	u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
	v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
	u3	0	-26250	-16406.3	26250	16406.25	0	0	0
	v3	-26250	0	26250	-49218.8	0	49218.75	0	0
	u4	0	0	0	0	0	0	0	0
	v4	0	0	0	0	0	0	0	0

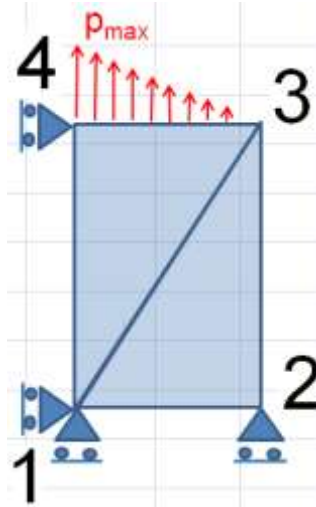
extended stiffness matrix of element 2:

		u1	v1	u2	v2	u3	v3	u4	v4
k_2^*	u1	16406.25	0	0	0	0	-26250	-16406.3	26250
	v1	0	49218.75	0	0	-26250	0	26250	-49218.75
	u2	0	0	0	0	0	0	0	0
	v2	0	0	0	0	0	0	0	0
	u3	0	-26250	0	0	126000	0	-126000	26250
	v3	-26250	0	0	0	0	42000	26250	-42000
	u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
	v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

global stiffness matrix:

		u1	v1	u2	v2	u3	v3	u4	v4
K	u1	142406.3	0	-126000	26250	0	-52500	-16406.3	26250
	v1	0	91218.75	26250	-42000	-52500	0	26250	-49218.75
	u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
	v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
	u3	0	-52500	-16406.3	26250	142406.3	0	-126000	26250
	v3	-52500	0	26250	-49218.8	0	91218.75	26250	-42000
	u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
	v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

Introduction of boundary conditions to the global stiffness matrix

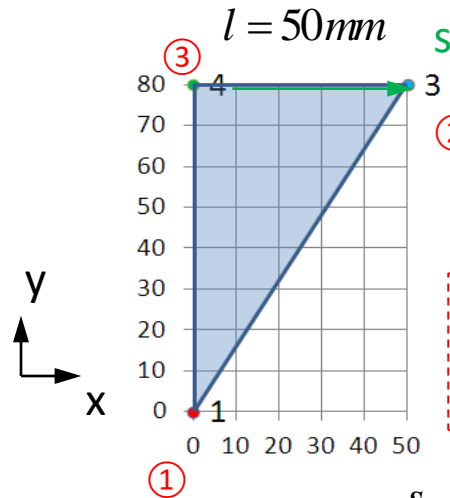
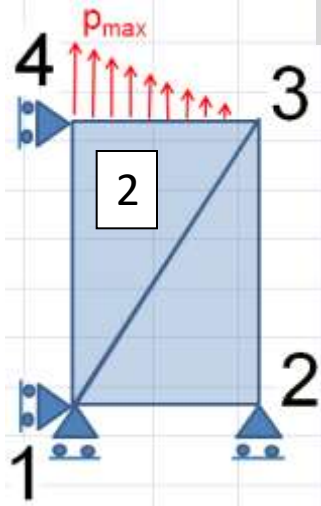


Boundary conditions



		u1	v1	u2	v2	u3	v3	u4	v4	
K= NDOF × NDOF	u1	142406.3	0	-126000	26250	0	-52500	-16406.3	26250	u1 = 0
	v1	0	91218.75	26250	-42000	-52500	0	26250	-49218.75	v1 = 0
	u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0	
	v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0	v2 = 0
	u3	0	-52500	-16406.3	26250	142406.3	0	-126000	26250	
	v3	-52500	0	26250	-49218.8	0	91218.75	26250	-42000	
	u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500	u4 = 0
	v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75	

Equivalent load vector of surface loads

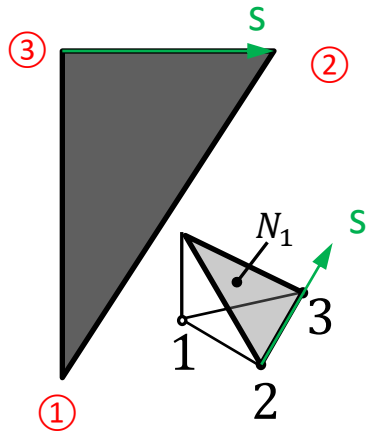


$$[p] = [p_x, p_y] = [0, p(s)] = \left[0, p_{\max} \left(1 - \frac{s}{l}\right) \right]$$

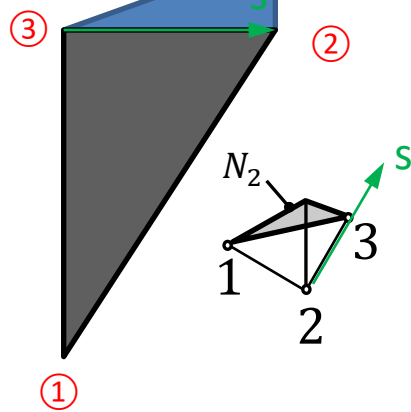
$$p_{\max} = 60 \text{ MPa}$$

$$[F^p]_e = t_e \int_0^l [p_x, p_y] \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} ds$$

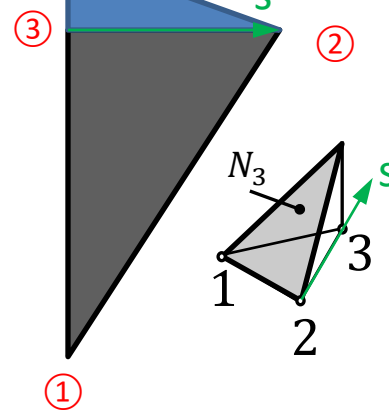
$$N_1(s) = 0$$



$$N_2(s) = \frac{s}{l}$$



$$N_3(s) = 1 - \frac{s}{l}$$



$$[F^p]_2 = h \int_0^l [p_x(s), p_y(s)] \begin{bmatrix} N_1(s) & 0 & N_2(s) & 0 & N_3(s) & 0 \\ 0 & N_1(s) & 0 & N_2(s) & 0 & N_3(s) \end{bmatrix} ds =$$

$$= [F_1^p, F_2^p, F_3^p, F_4^p, F_5^p, F_6^p]_2$$

Equivalent load vector of surface loads

$$F_1^p = h \int_0^l p_x(s) N_1(s) ds = h \int_0^l 0 \cdot 0 ds = 0$$

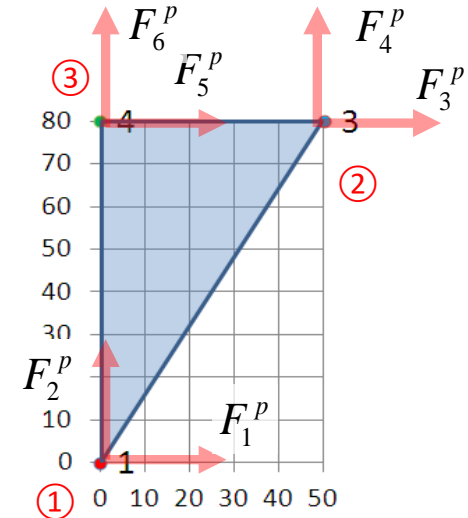
$$F_2^p = h \int_0^l p_y(s) N_1(s) ds = h \int_0^l p_{\max} \left(1 - \frac{s}{l}\right) \cdot 0 ds = 0$$

$$F_3^p = h \int_0^l p_x(s) N_2(s) ds = h \int_0^l 0 \cdot \frac{s}{l_1} ds = 0$$

$$F_4^p = h \int_0^l p_y(s) N_2(s) ds = h \int_0^l p_{\max} \left(1 - \frac{s}{l}\right) \cdot \frac{s}{l} ds = \frac{1}{6} p_{\max} lh = 1000 N$$

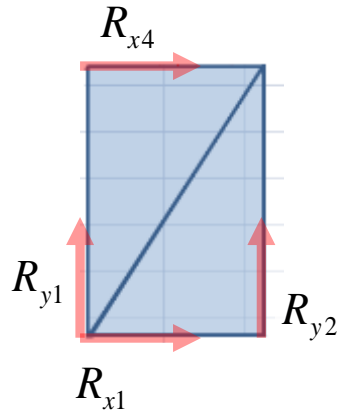
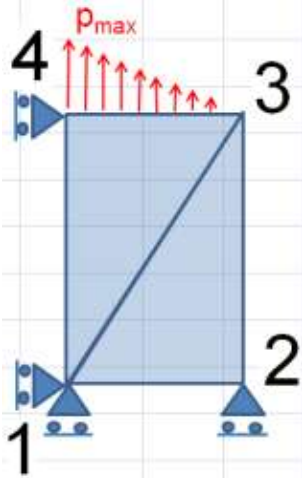
$$F_5^p = h \int_0^l p_x(s) N_3(s) ds = h \int_0^l 0 \cdot \left(1 - \frac{s}{l}\right) ds = 0$$

$$F_6^p = h \int_0^l p_y(s) N_3(s) ds = h \int_0^l p_{\max} \left(1 - \frac{s}{l}\right) \cdot \left(1 - \frac{s}{l}\right) ds = \frac{1}{3} p_{\max} lh = 2000 N$$



	0
	0
	0
F^e	0
	0
	1000
	0
	2000

Determination of nodal displacements



	0		Rx1		Rx1
	0		Ry1		Ry1
	0		0		0
$F^e =$	0		$F^n =$ Ry2		$F =$ Ry2
	0		0		0
	1000		0		1000
	0		Rx4		Rx4
	2000		0		2000

NDOF × 1

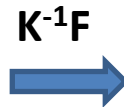
		u2	u3	v3	v4
$K =$	u2	142406.3	-16406.3	26250	0
	u3	-16406.3	142406.3	0	26250
	v3	26250	0	91218.75	-42000
	v4	0	26250	-42000	91218.75

$N \times N$

		u2	u3	v3	v4
$K^{-1} =$	u2	7.71864E-06	1.20993E-06	-3.02221E-06	-1.7397E-06
	u3	1.20993E-06	7.71864E-06	-1.7397E-06	-3.02221E-06
	v3	-3.02221E-06	-1.7397E-06	1.53082E-05	7.54899E-06
	v4	-1.7397E-06	-3.02221E-06	7.54899E-06	1.53082E-05

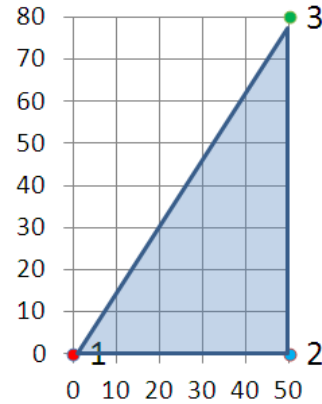
$N \times N$

	0
$F =$	0
$N \times 1$	1000
	2000



	-0.006502	mm	u2
$q =$	-0.007784	mm	u3
$N \times 1$	0.030406	mm	v3
	0.038165	mm	v4

Determination of strain and stress in element 1



$$B_1 = \begin{bmatrix} -0.02 & 0 & 0.02 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0125 & 0 & 0.0125 \\ 0 & -0.02 & -0.0125 & 0.02 & 0.0125 & 0 \end{bmatrix}$$

$$q_1 = \begin{matrix} 0 \text{ mm} & u1 \\ 0 \text{ mm} & v1 \\ -0.006502 \text{ mm} & u2 \\ 0 \text{ mm} & v2 \\ -0.007784 \text{ mm} & u3 \\ 0.030406 \text{ mm} & v3 \end{matrix}$$

$B_1 q_1$

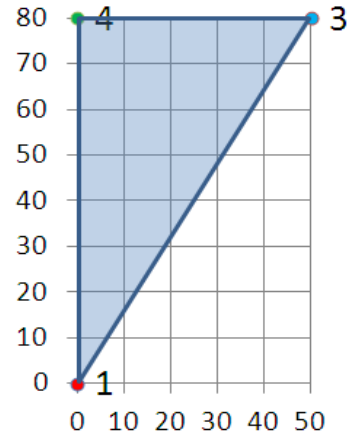
$$D = \begin{bmatrix} 78750 & 26250 & 0 \\ 26250 & 78750 & 0 \\ 0 & 0 & 26250 \end{bmatrix}$$

$$\epsilon_1 = \begin{bmatrix} -0.000130032 \\ 0.000380077 \\ -1.60313E-05 \end{bmatrix}$$

$D \epsilon_1$

$$\sigma_1 = \begin{bmatrix} -0.263 \text{ MPa} \\ 26.52 \text{ MPa} \\ -0.421 \text{ MPa} \end{bmatrix}$$

Determination of strain and stress in element 2



$$B_2 = \begin{bmatrix} 0 & 0 & 0.02 & 0 & -0.02 & 0 \\ 0 & -0.0125 & 0 & 0 & 0 & 0.0125 \\ -0.0125 & 0 & 0 & 0.02 & 0.0125 & -0.02 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0 \text{ mm} & u_1 \\ 0 \text{ mm} & v_1 \\ -0.007784 \text{ mm} & u_3 \\ 0.030406 \text{ mm} & v_3 \\ 0 \text{ mm} & u_4 \\ 0.038165 \text{ mm} & v_4 \end{bmatrix}$$

$$B_2 q_2$$

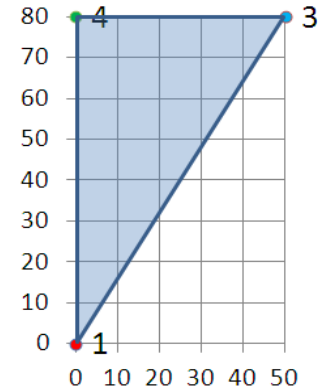
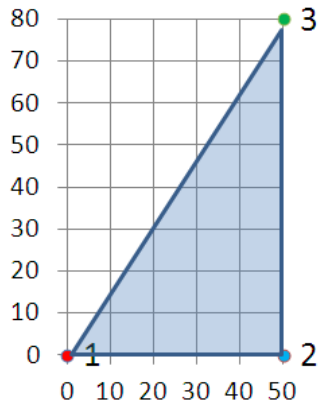
$$D = \begin{bmatrix} 78750 & 26250 & 0 \\ 26250 & 78750 & 0 \\ 0 & 0 & 26250 \end{bmatrix}$$

$$\epsilon_2 = \begin{bmatrix} -0.000155682 \\ 0.000477066 \\ -0.000155183 \end{bmatrix}$$

$$D \epsilon_2$$

$$\sigma_2 = \begin{bmatrix} 0.263 \text{ MPa} \\ 33.48 \text{ MPa} \\ -4.074 \text{ MPa} \end{bmatrix}$$

Determination of elastic strain energy in elements



$$U_e = \frac{1}{2} \int_{\Omega_e} [\varepsilon] \{\sigma\} d\Omega_e = \frac{1}{2} [\varepsilon] \{\sigma\} \int_{\Omega_e} d\Omega_e$$

$$U_1 = 20.23940803 \text{ Nmm}$$

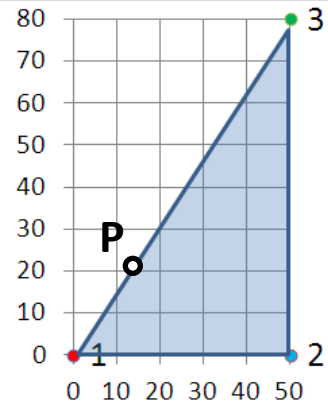
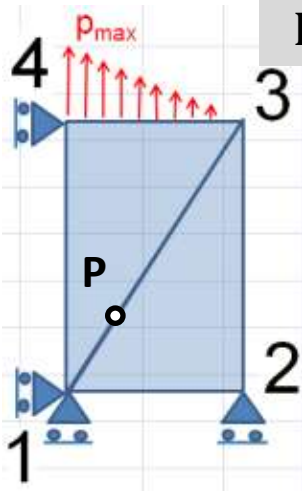
$$U_2 = 33.12895375 \text{ Nmm}$$

$$U = 53.37 \text{ Nmm}$$

$$U_{\text{exact}} = 59.35 \text{ Nmm}$$

$$U = 89.93\% U_{\text{exact}}$$

Determination of solutions at point P on the boundary of elements



Element 1

node	x _i	y _i	x _j	y _j	x _k	y _k	a _i	b _i	c _i
1	0	0	50	0	50	80	4000	-80	0
2	50	0	50	80	0	0	0	80	-50
3	50	80	0	0	50	0	0	0	50

$$N_1(x_P, y_P) = N_1(12.5, 20) = \frac{a_1 + b_1 x_P + c_1 y_P}{2 \cdot A_e} = \frac{4000 \text{ mm}^2 + (-80 \text{ mm}) \cdot 12.5 \text{ mm} + 0 \text{ mm} \cdot 20 \text{ mm}}{2 \cdot 2000 \text{ mm}^2} = \frac{3}{4}$$

$$N_2(x_P, y_P) = N_2(12.5, 20) = \frac{a_2 + b_2 x_P + c_2 y_P}{2 \cdot A_e} = \frac{0 + 80 \cdot 12.5 + (-50) \cdot 20}{2 \cdot 2000} = 0$$

$$N_3(x_P, y_P) = N_3(12.5, 20) = \frac{a_3 + b_3 x_P + c_3 y_P}{2 \cdot A_e} = \frac{0 + 0 \cdot 12.5 + 50 \cdot 20}{2 \cdot 2000} = \frac{1}{4}$$

Determination of solutions at point P on the boundary of elements

$$N_1(12.5,20) + N_2(12.5,20) + N_3(12.5,20) = \frac{3}{4} + 0 + \frac{1}{4} = 1$$

$$x = \sum_{i=1}^3 N_i(x, y) \cdot x_i \Rightarrow x_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot x_i = N_1 \cdot x_1 + N_2 \cdot x_2 + N_3 \cdot x_3 =$$

$$= \frac{3}{4} \cdot 0 + 0 \cdot 50 + \frac{1}{4} \cdot 50 = 12.5 \text{ mm}$$

$$y = \sum_{i=1}^3 N_i(x, y) \cdot y_i \Rightarrow y_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot y_i = N_1 \cdot y_1 + N_2 \cdot y_2 + N_3 \cdot y_3 =$$

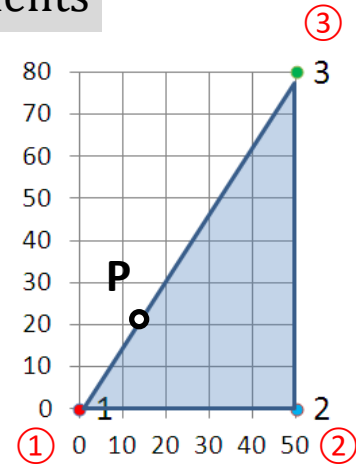
$$= \frac{3}{4} \cdot 0 + 0 \cdot 0 + \frac{1}{4} \cdot 80 = 20 \text{ mm}$$

$$u = \sum_{i=1}^3 N_i(x, y) \cdot u_i \Rightarrow u_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot u_i = N_1 \cdot u_1 + N_2 \cdot u_2 + N_3 \cdot u_3 =$$

$$= \frac{3}{4} \cdot 0 + 0 \cdot (-0.006502) + \frac{1}{4} \cdot (-0.007784) = -0.00195 \text{ mm}$$

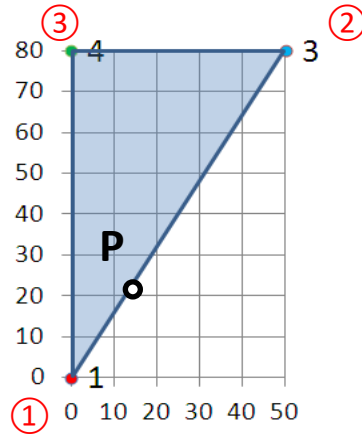
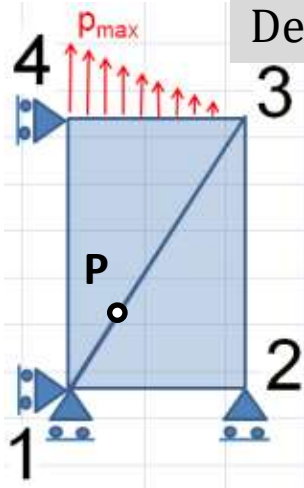
$$v = \sum_{i=1}^3 N_i(x, y) \cdot v_i \Rightarrow v_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot v_i = N_1 \cdot v_1 + N_2 \cdot v_2 + N_3 \cdot v_3 =$$

$$= \frac{3}{4} \cdot 0 + 0 \cdot 0 + \frac{1}{4} \cdot 0.030406 = 0.0076 \text{ mm}$$



	0 mm	u1
	0 mm	v1
q₁	-0.006502 mm	u2
ne × 1	0 mm	v2
	-0.007784 mm	u3
	0.030406 mm	v3

Determination of solutions at point P on the boundary of elements



Element 2

node	x_i	y_i	x_j	y_j	x_k	y_k	a_i	b_i	c_i
1	0	0	50	80	0	80	4000	0	-50
3	50	80	0	80	0	0	0	80	0
4	0	80	0	0	50	80	0	-80	50

$$N_1(x_P, y_P) = N_1(12.5, 20) = \frac{a_1 + b_1 x_P + c_1 y_P}{2 \cdot A_e} = \frac{4000 \text{ mm}^2 + 0 \text{ mm} \cdot 12.5 \text{ mm} + (-50 \text{ mm}) \cdot 20 \text{ mm}}{2 \cdot 2000 \text{ mm}^2} = \frac{3}{4}$$

$$N_2(x_P, y_P) = N_2(12.5, 20) = \frac{a_2 + b_2 x_P + c_2 y_P}{2 \cdot A_e} = \frac{0 + 80 \cdot 12.5 + 0 \cdot 20}{2 \cdot 2000} = \frac{1}{4}$$

$$N_3(x_P, y_P) = N_3(12.5, 20) = \frac{a_3 + b_3 x_P + c_3 y_P}{2 \cdot A_e} = \frac{0 + (-80) \cdot 12.5 + 50 \cdot 20}{2 \cdot 2000} = 0$$

Determination of solutions at point P on the boundary of elements

$$N_1(12.5,20) + N_2(12.5,20) + N_3(12.5,20) = \frac{3}{4} + \frac{1}{4} + 0 = 1$$

$$x = \sum_{i=1}^3 N_i(x, y) \cdot x_i \Rightarrow x_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot x_i = N_1 \cdot x_1 + N_2 \cdot x_2 + N_3 \cdot x_3 =$$

$$= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 50 + 0 \cdot 0 = 12.5 \text{ mm}$$

$$y = \sum_{i=1}^3 N_i(x, y) \cdot y_i \Rightarrow y_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot y_i = N_1 \cdot y_1 + N_2 \cdot y_2 + N_3 \cdot y_3 =$$

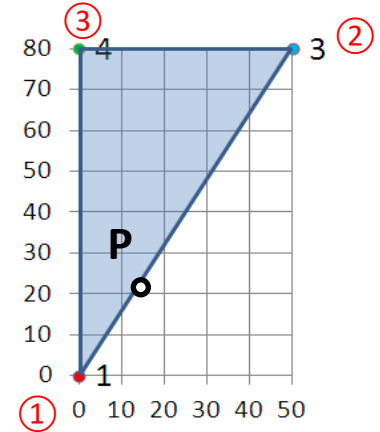
$$= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 80 + 0 \cdot 80 = 20 \text{ mm}$$

$$u = \sum_{i=1}^3 N_i(x, y) \cdot u_i \Rightarrow u_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot u_i = N_1 \cdot u_1 + N_2 \cdot u_2 + N_3 \cdot u_3 =$$

$$= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot (-0.007784) + 0 \cdot 0 = -0.00195 \text{ mm}$$

$$v = \sum_{i=1}^3 N_i(x, y) \cdot v_i \Rightarrow v_p = \sum_{i=1}^3 N_i(x_p, y_p) \cdot v_i = N_1 \cdot v_1 + N_2 \cdot v_2 + N_3 \cdot v_3 =$$

$$= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 0.030406 + 0 \cdot 0.038165 = 0.0076 \text{ mm}$$



local notation

global notation

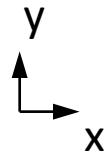
global notation

	0 mm	u1
	0 mm	v1
$q_2 =$	-0.007784 mm	u3
$ne \times 1$	0.030406 mm	v3
	0 mm	u4
	0.038165 mm	v4

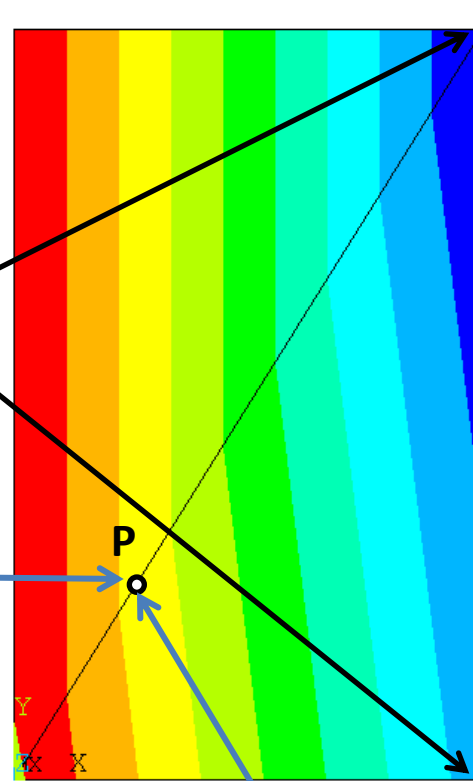
Displacements at point P on the boundary of elements

UX displacement

	-0.006502	mm	u2
q=	-0.007784	mm	u3
N × 1	0.030406	mm	v3
	0.038165	mm	v4



-0.00195



PLOT NO. 1
 NODAL SOLUTION
 STEP=1
 SUB =1
 TIME=1
 UX (AVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 AVRES=Mat
 DMX =.038165
 SMN =-.007784
 -.007784
 -.006919
 -.006054
 -.005189
 -.004324
 -.00346
 -.002595
 -.00173
 -.865E-03
 0

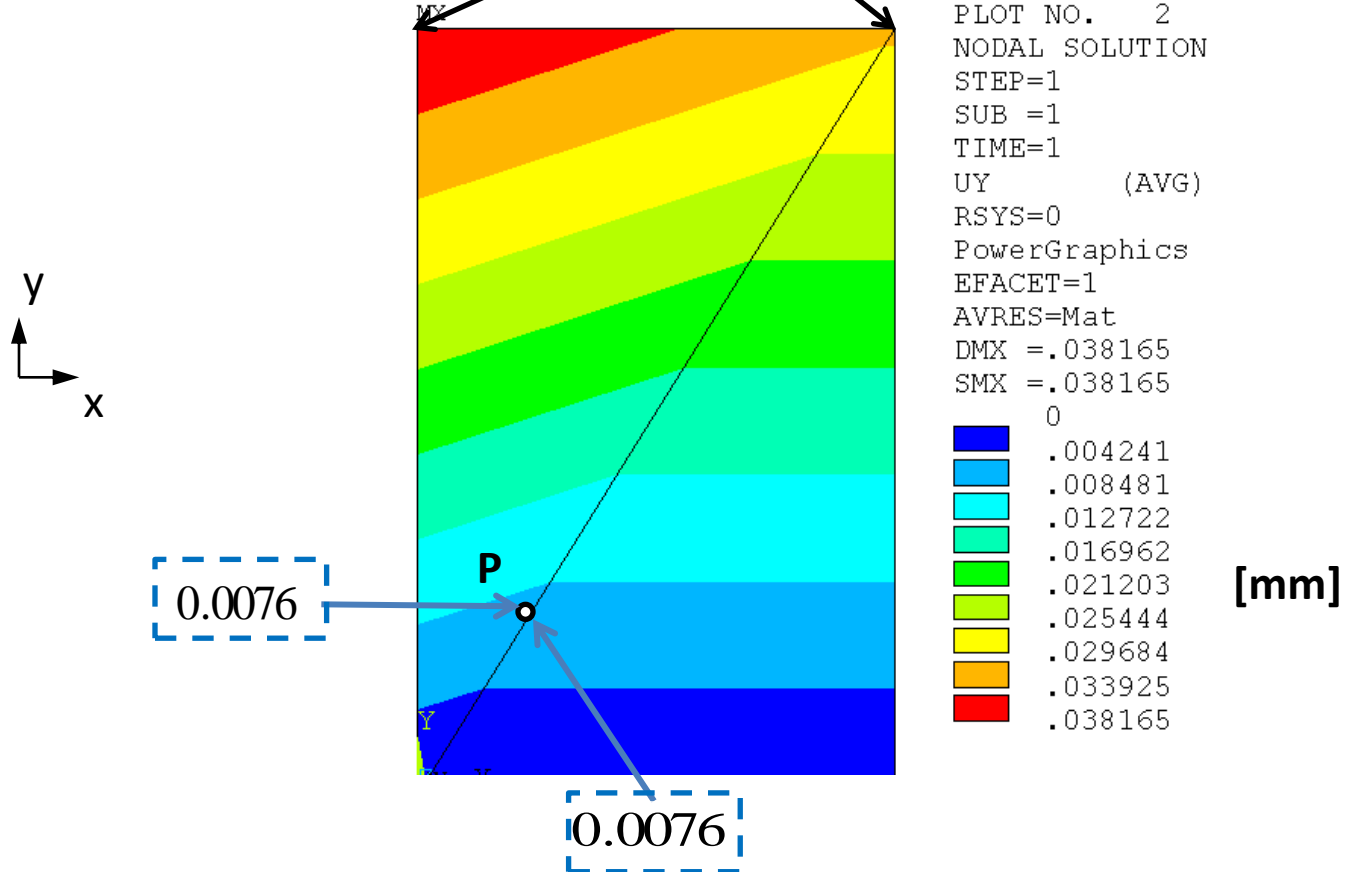
-0.00195

[mm]

Displacements at point P on the boundary of elements

UY displacement

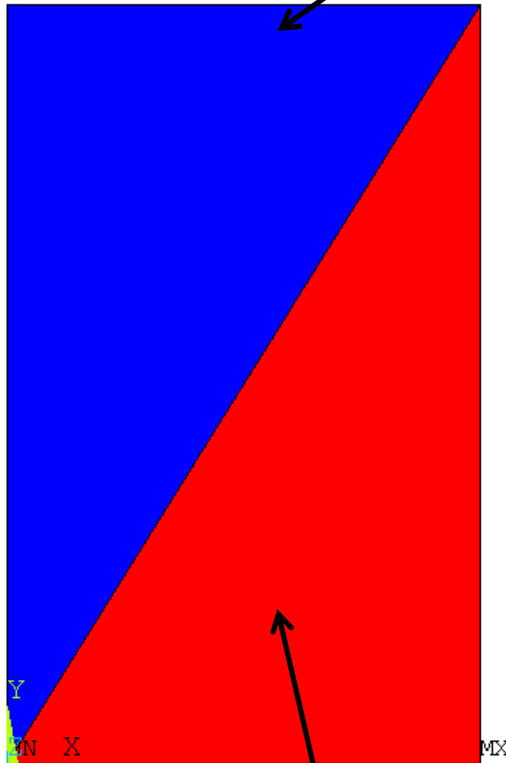
	-0.006502	mm	u2
q=	-0.007784	mm	u3
N x 1	0.030406	mm	v3
	0.038165	mm	v4



Strain in X at the boundary of elements



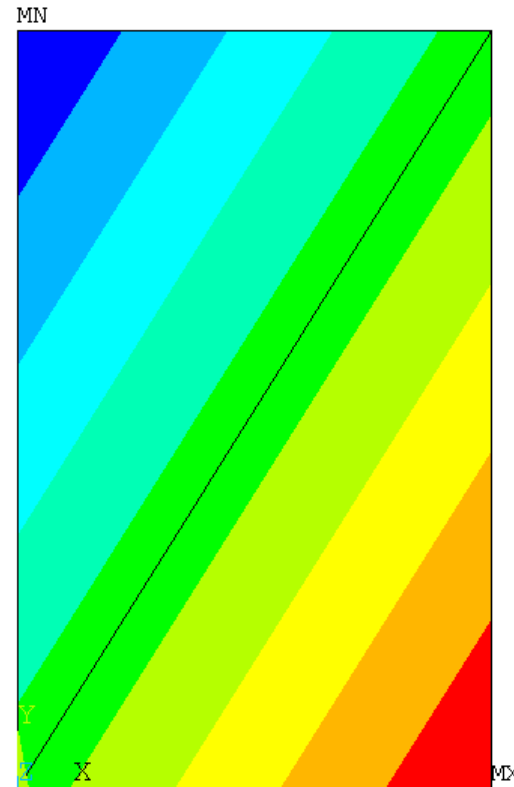
$\epsilon_2 =$	-0.000155682
	0.000477066
	-0.000155183



ELEMENT SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELX (NOAVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 DMX =.038165
 SMN =-.156E-03
 SMX =-.130E-03

Blue	-.156E-03
Light Blue	-.153E-03
Cyan	-.150E-03
Light Green	-.147E-03
Green	-.144E-03
Light Green	-.141E-03
Yellow	-.139E-03
Orange	-.136E-03
Red	-.133E-03
Dark Red	-.130E-03

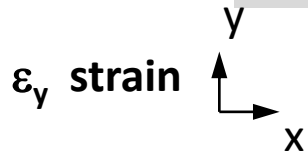
$\epsilon_1 =$	-0.000130032
	0.000380077
	-1.60313E-05



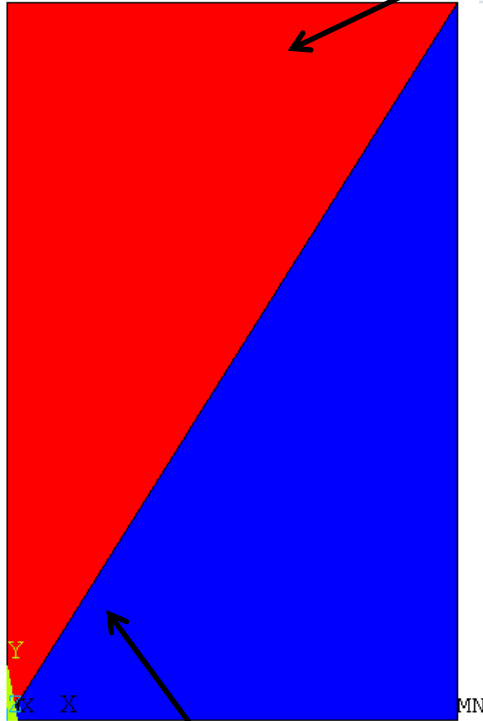
PLOT NO. 12
 NODAL SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELX (AVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 AVRES=Mat
 DMX =.038165
 SMN =-.156E-03
 SMX =-.130E-03

Blue	-.156E-03
Light Blue	-.153E-03
Cyan	-.150E-03
Light Green	-.147E-03
Green	-.144E-03
Light Green	-.141E-03
Yellow	-.139E-03
Orange	-.136E-03
Red	-.133E-03
Dark Red	-.130E-03

Strain in Y at the boundary of elements

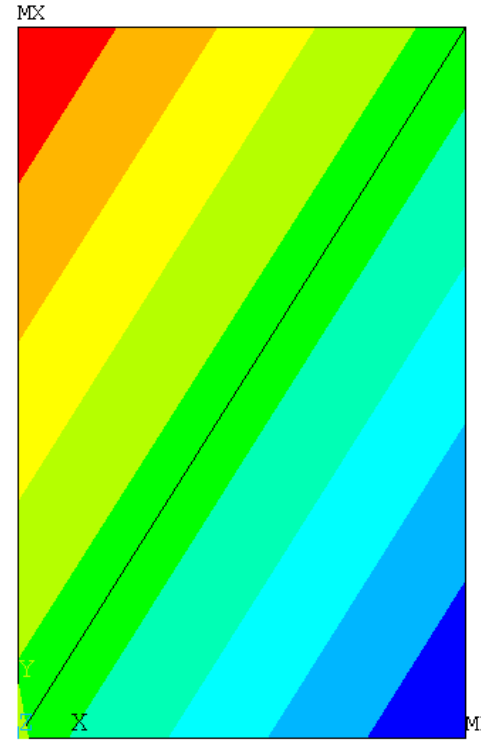


$\epsilon_2 =$	-0.000155682
	0.000477066
	-0.000155183



ELEMENT SOLUTION

STEP=1
SUB =1
TIME=1
EPELY (NOAVG)
RSYS=0
PowerGraphics
EFACET=1
DMX =.038165
SMN =.380E-03
SMX =.477E-03
.380E-03
.391E-03
.402E-03
.412E-03
.423E-03
.434E-03
.445E-03
.456E-03
.466E-03
.477E-03



PLOT NO. 13

NODAL SOLUTION

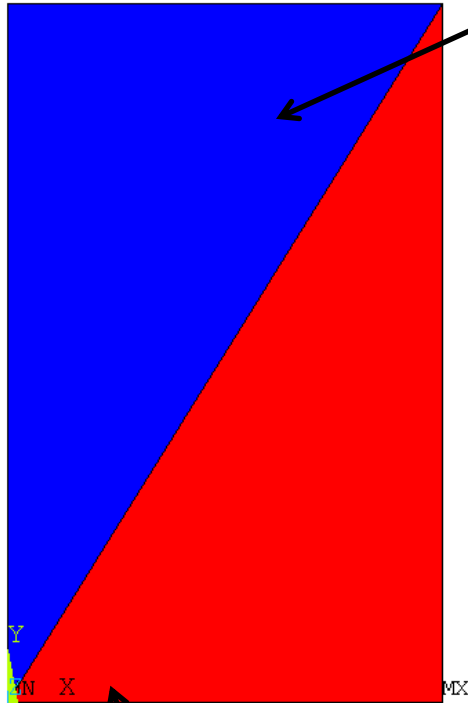
STEP=1
SUB =1
TIME=1
EPELY (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.038165
SMN =.380E-03
SMX =.477E-03
.380E-03
.391E-03
.402E-03
.412E-03
.423E-03
.434E-03
.445E-03
.456E-03
.466E-03
.477E-03

$\epsilon_1 =$	-0.000130032
	0.000380077
	-1.60313E-05

Shear strain at the boundary of elements

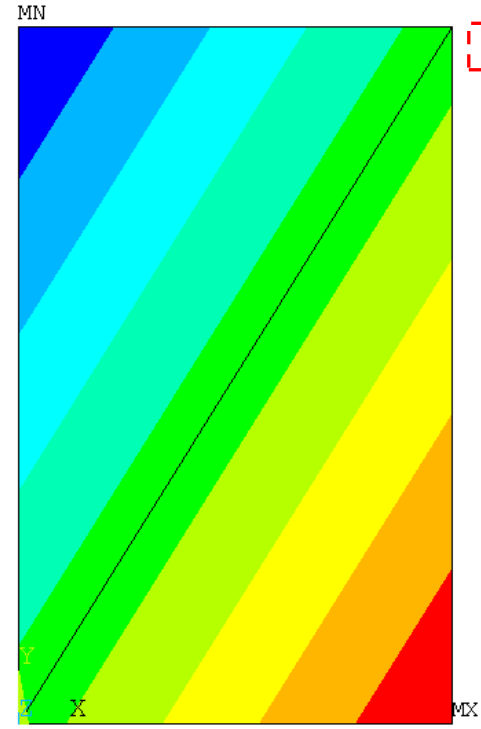


$\epsilon_2 =$	-0.000155682
	0.000477066
	-0.000155183



ELEMENT SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELXY (NOAVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 DMX =.038165
 SMN =-.155E-03
 SMX =-.160E-04

Blue	-.155E-03
Light Blue	-.140E-03
Cyan	-.124E-03
Light Green	-.109E-03
Green	-.933E-04
Yellow-Green	-.779E-04
Yellow	-.624E-04
Orange	-.470E-04
Red-Orange	-.315E-04
Red	-.160E-04



PLOT NO. 14
 NODAL SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELXY (AVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 AVRES=Mat
 DMX =.038165
 SMN =-.155E-03
 SMX =-.160E-04

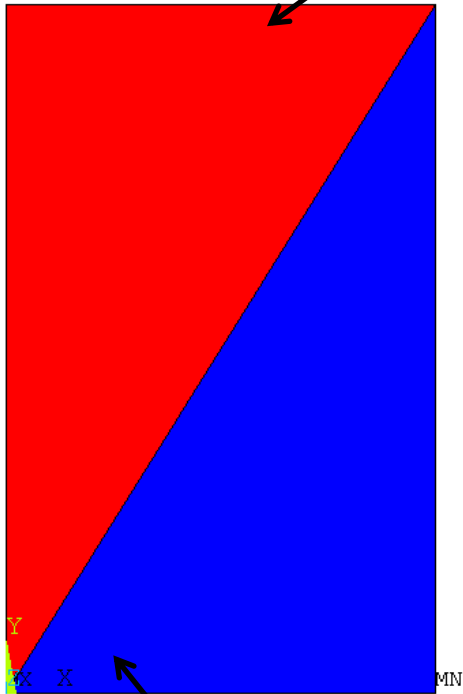
Blue	-.155E-03
Light Blue	-.140E-03
Cyan	-.124E-03
Light Green	-.109E-03
Green	-.933E-04
Yellow-Green	-.779E-04
Yellow	-.624E-04
Orange	-.470E-04
Red-Orange	-.315E-04
Red	-.160E-04

$\epsilon_1 =$	-0.000130032
	0.000380077
	-1.60313E-05

Stress in X at the boundary of elements



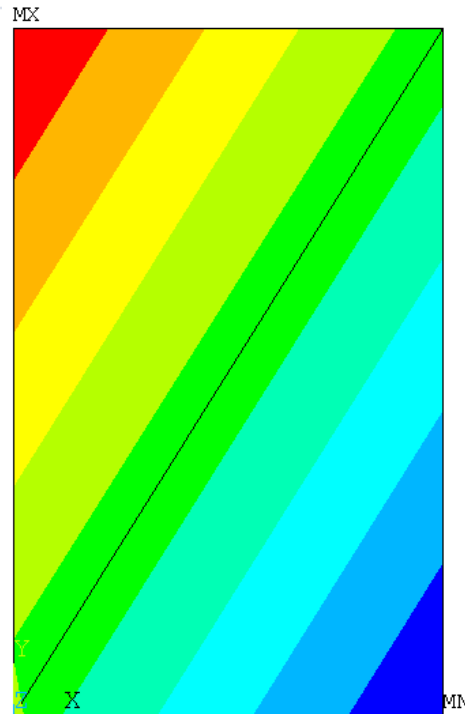
$\sigma_2 =$	0.263	MPa
	33.48	MPa
	-4.074	MPa



ELEMENT SOLUTION

STEP=1
SUB =1
TIME=1
SX (NOAVG)
RSYS=0
PowerGraphics
EFACET=1
DMX =.038165
SMN =-.263011
SMX =.263011

Blue	-.263011
Light Blue	-.204564
Cyan	-.146117
Light Green	-.08767
Green	-.029223
Yellow-Green	.029223
Yellow	.08767
Orange	.146117
Light Orange	.204564
Red	.263011



PLOT NO. 9

NODAL SOLUTION

STEP=1
SUB =1
TIME=1
SX (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.038165
SMN =-.263011
SMX =.263011

Blue	-.263011
Light Blue	-.204564
Cyan	-.146117
Light Green	-.08767
Green	-.029223
Yellow-Green	.029223
Yellow	.08767
Orange	.146117
Light Orange	.204564
Red	.263011

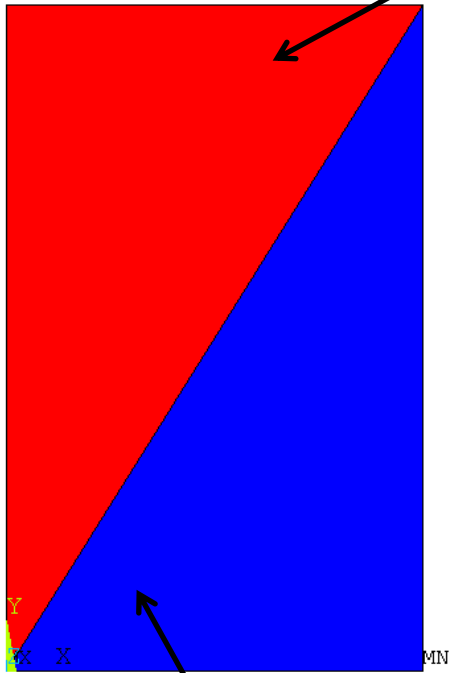
$\sigma_1 =$	-0.263	MPa
	26.52	MPa
	-0.421	MPa

[MPa]

Stress in Y at the boundary of elements



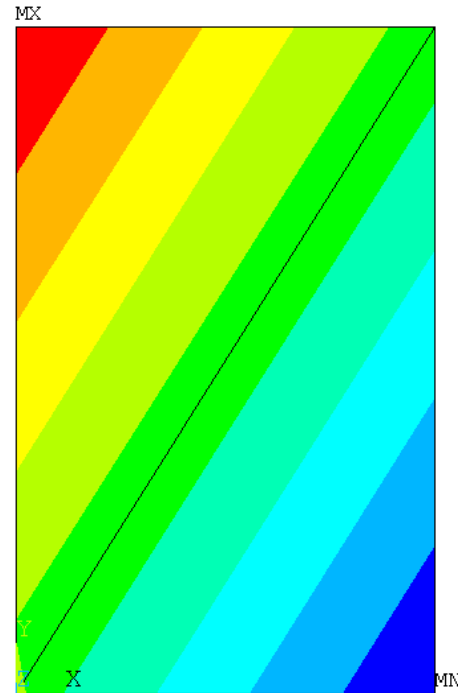
$\sigma_2 =$	0.263	MPa
	33.48	MPa
	-4.074	MPa



ELEMENT SOLUTION

STEP=1
SUB =1
TIME=1
SY (NOAVG)
RSYS=0
PowerGraphics
EFACET=1
DMX =.038165
SMN =26.518
SMX =33.482

26.518
27.292
28.065
28.839
29.613
30.387
31.161
31.935
32.708
33.482



PLOT NO = 10

NODAL SOLUTION

STEP=1
SUB =1
TIME=1
SY (AVG)
RSYS=0
PowerGraphics
EFACET=1
AVRES=Mat
DMX =.038165
SMN =26.518
SMX =33.482

26.518
27.292
28.065
28.839
29.613
30.387
31.161
31.935
32.708
33.482

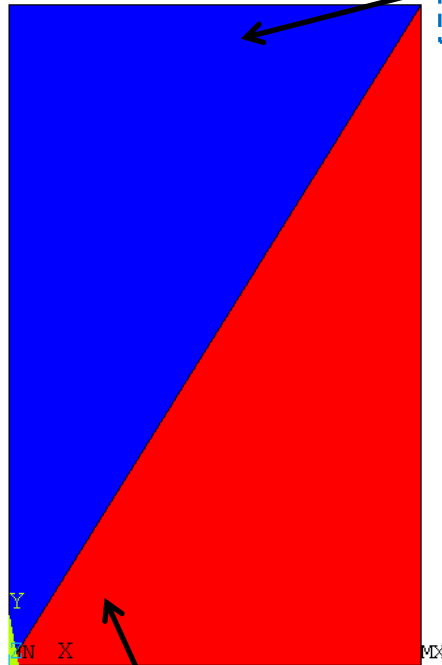
$\sigma_1 =$	-0.263	MPa
	26.52	MPa
	-0.421	MPa

[MPa]

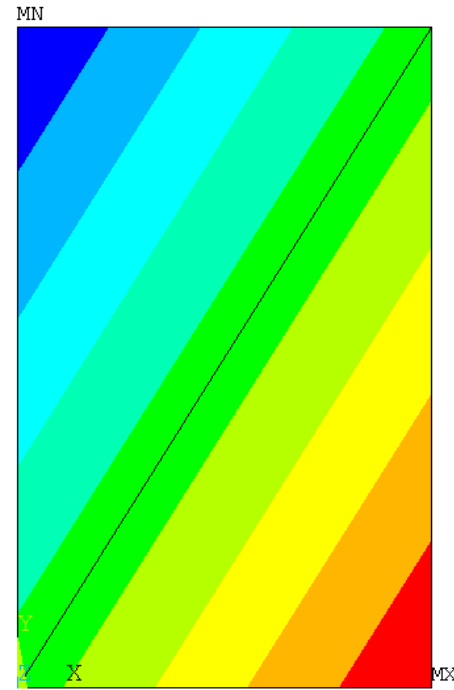
Shear stress at the boundary of elements



$\sigma_2 =$	0.263	MPa
	33.48	MPa
	-4.074	MPa



ELEMENT SOLUTION	
STEP=1	
SUB =1	
TIME=1	
SXY (NOAVG)	
RSYS=0	
PowerGraphics	
EFACET=1	
DMX =.038165	
SMN =-4.074	
SMX =-.420818	
-4.074	
-3.668	
-3.262	
-2.856	
-2.45	
-2.044	
-1.638	
-1.233	
-.826679	
-.420818	



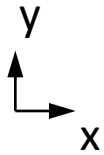
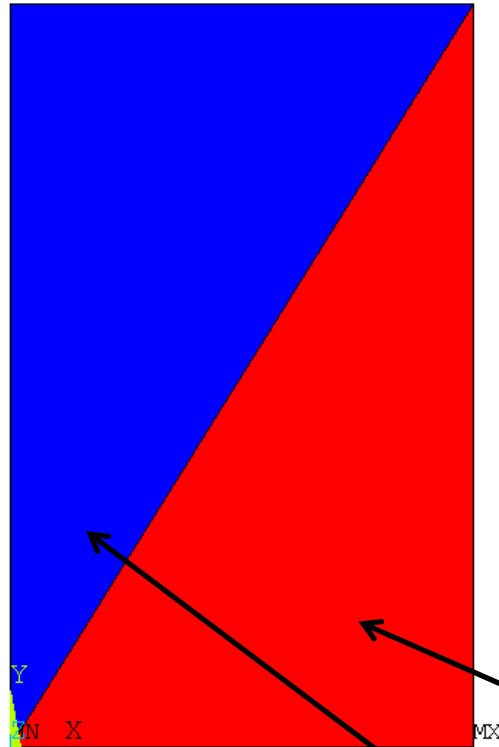
PLOT NO. 11	
NODAL SOLUTION	
STEP=1	
SUB =1	
TIME=1	
SXY (AVG)	
RSYS=0	
PowerGraphics	
EFACET=1	
AVRES=Mat	
DMX =.038165	
SMN =-4.074	
SMX =-.420818	
-4.074	
-3.668	
-3.262	
-2.856	
-2.45	
-2.044	
-1.638	
-1.233	
-.826679	
-.420818	

$\sigma_1 =$	-0.263	MPa
	26.52	MPa
	-0.421	MPa

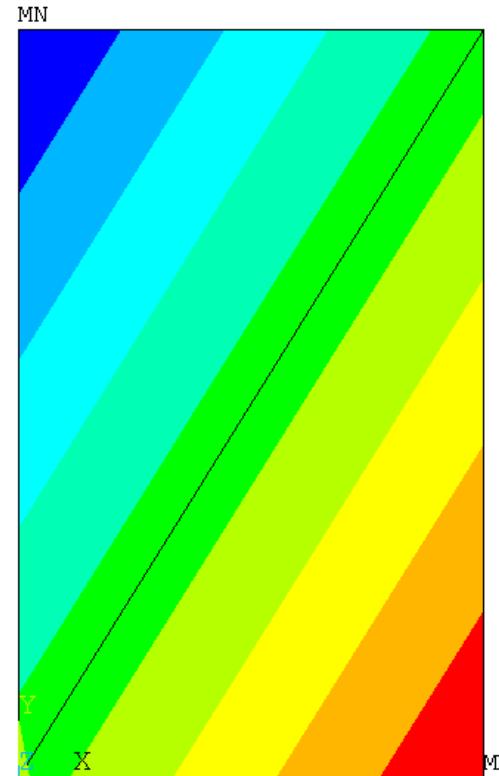
[MPa]

Strain in Z at the boundary of elements

ϵ_z strain

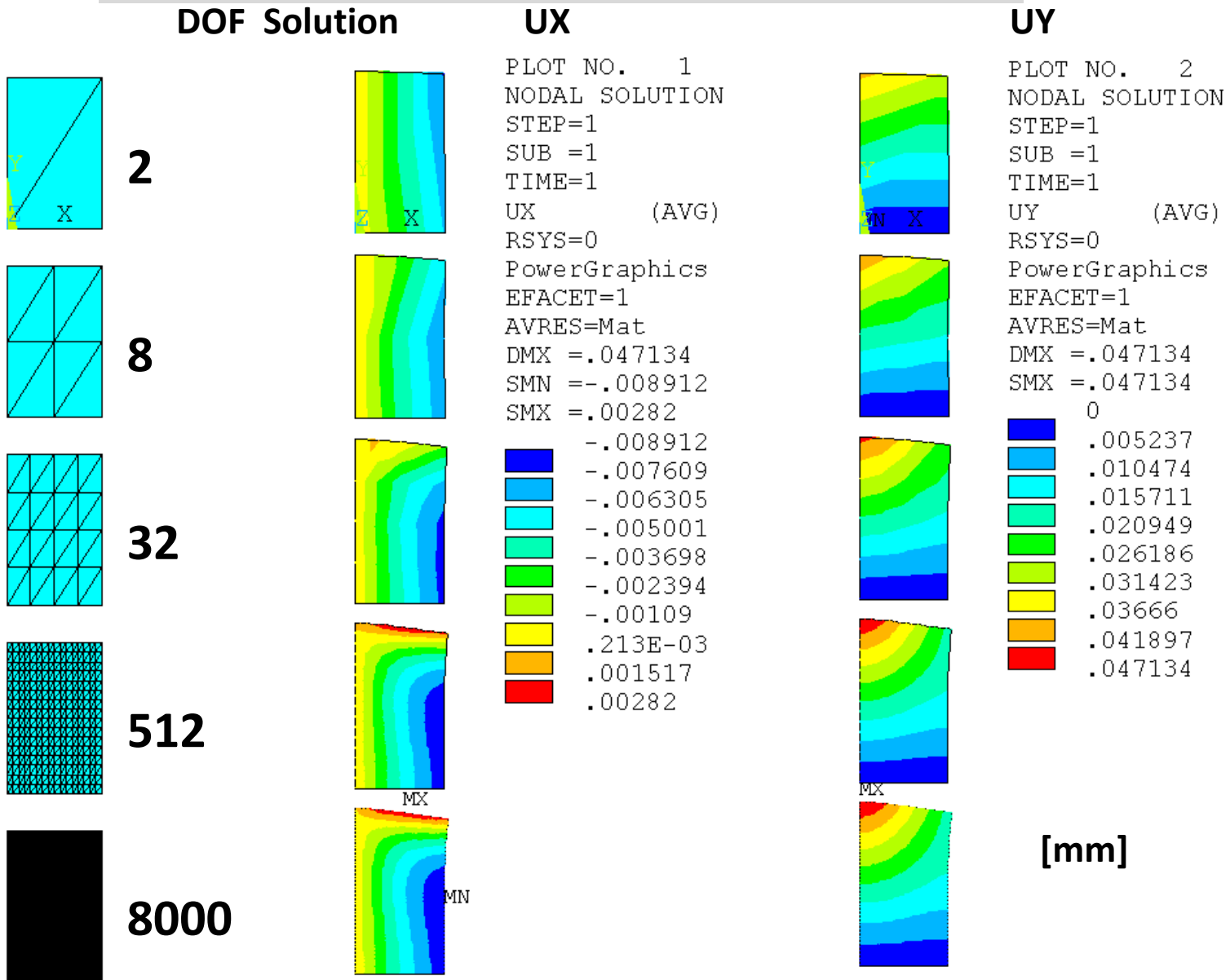
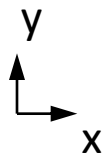
PLOT NO. 6
 ELEMENT SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELZ (NOAVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 DMX =.038165
 SMN =-.161E-03
 SMX =-.125E-03
 -.161E-03
 -.157E-03
 -.153E-03
 -.149E-03
 -.145E-03
 -.141E-03
 -.137E-03
 -.133E-03
 -.129E-03
 -.125E-03



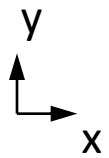
PLOT NO. 15
 NODAL SOLUTION
 STEP=1
 SUB =1
 TIME=1
 EPELZ (AVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 AVRES=Mat
 DMX =.038165
 SMN =-.161E-03
 SMX =-.125E-03
 -.161E-03
 -.157E-03
 -.153E-03
 -.149E-03
 -.145E-03
 -.141E-03
 -.137E-03
 -.133E-03
 -.129E-03
 -.125E-03

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = \begin{cases} -\frac{1}{3.7 \cdot 10^4}(-0.263 + 26.52) = -0.125 \cdot 10^{-3} \\ -\frac{1}{3.7 \cdot 10^4}(0.263 + 33.48) = -0.161 \cdot 10^{-3} \end{cases}$$

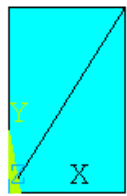
The impact of discretization on the quality of results



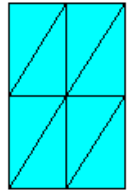
The impact of discretization on the quality of results



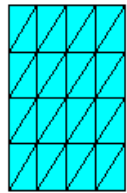
Horizontal stress σ_x



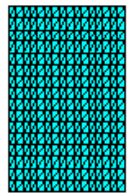
2



8



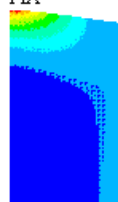
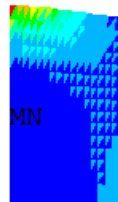
32



512



8000



PLOT NO. 3
 ELEMENT SOLUTION
 STEP=1
 SUB =1
 TIME=1
 SX (NOAVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 DMX =.047134
 SMN =-4.88
 SMX =35.587

Blue	-4.88
Light Blue	-.383303
Cyan	4.113
Light Green	8.609
Green	13.106
Yellow-Green	17.602
Yellow	22.098
Orange	26.595
Red-Orange	31.091
Red	35.587

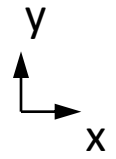


PLOT NO. 7
 NODAL SOLUTION
 STEP=1
 SUB =1
 TIME=1
 SX (AVG)
 RSYS=0
 PowerGraphics
 EFACET=1
 AVRES=Mat
 DMX =.047134
 SMN =-4.784
 SMX =35.587

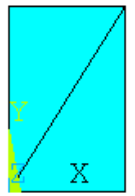
Blue	-4.784
Light Blue	-.298737
Cyan	4.187
Light Green	8.673
Green	13.159
Yellow-Green	17.644
Yellow	22.13
Orange	26.616
Red-Orange	31.102
Red	35.587

[MPa]

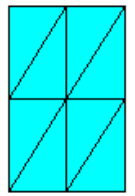
The impact of discretization on the quality of results



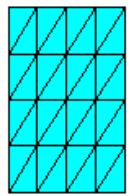
Vertical stress σ_y



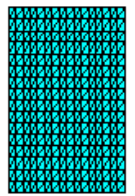
2



8



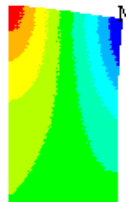
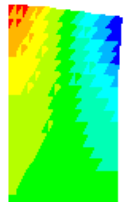
32



512



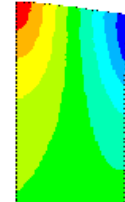
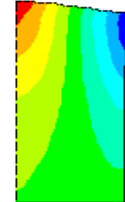
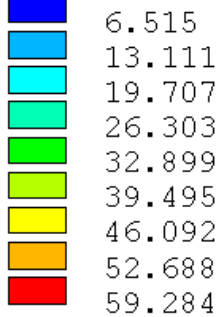
8000



PLOT NO. 4
ELEMENT SOLUTION

STEP=1
SUB =1
TIME=1
SY (NOAVG)
RSYS=0

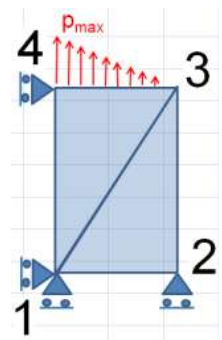
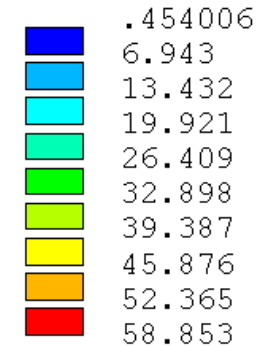
PowerGraphics
EFACET=1
DMX =.047134
SMN =-.081405
SMX =59.284



PLOT NO. 8
NODAL SOLUTION

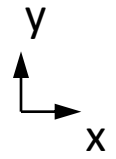
STEP=1
SUB =1
TIME=1
SY (AVG)
RSYS=0

PowerGraphics
EFACET=1
AVRES=Mat
DMX =.047134
SMN =.454006
SMX =58.853

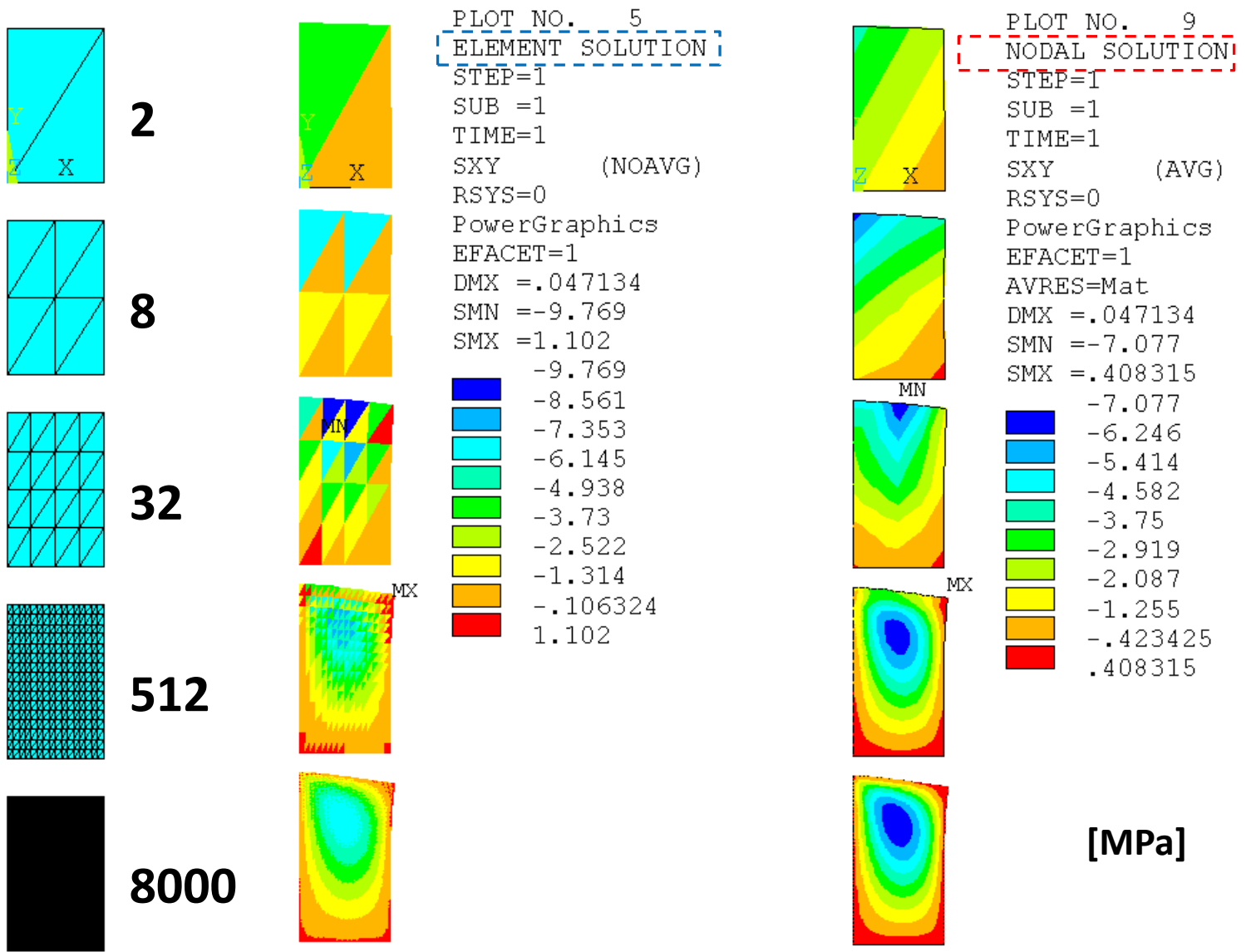


[MPa]

The impact of discretization on the quality of results



Shear stress τ_{xy}



The impact of discretization on the quality of results

