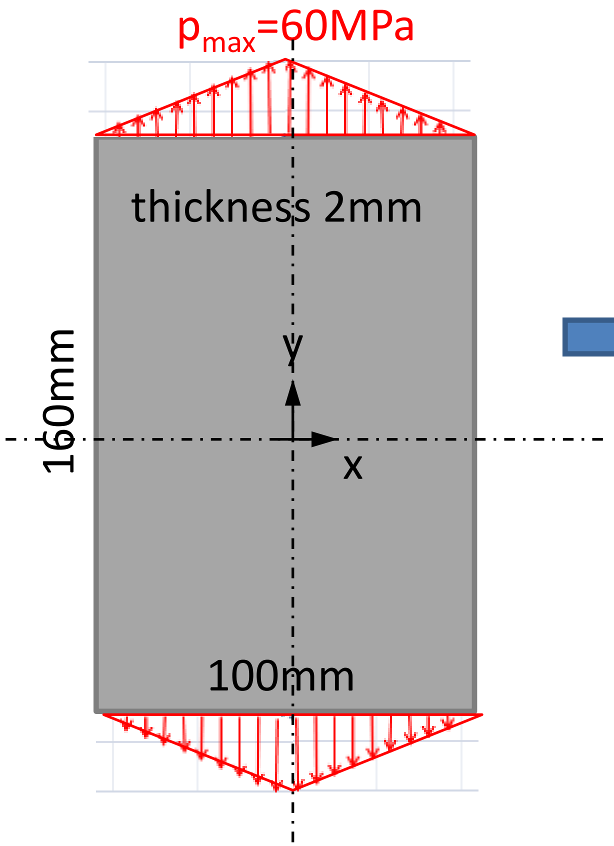
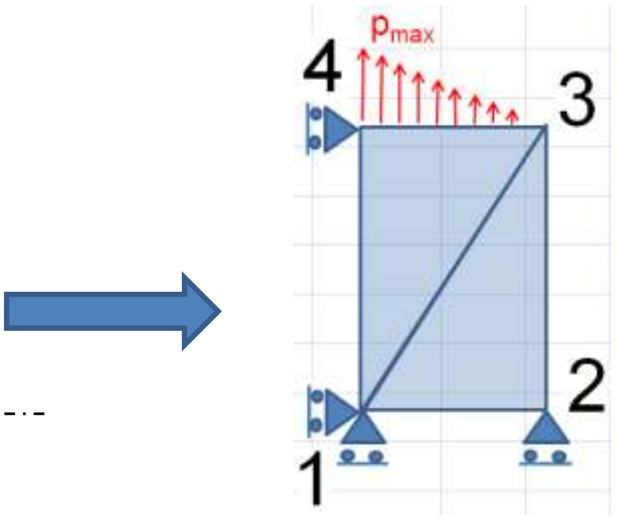
Finite element method (FEM1)

Lecture 3B. 2D Plate modeled using CST finite elements

03.2025

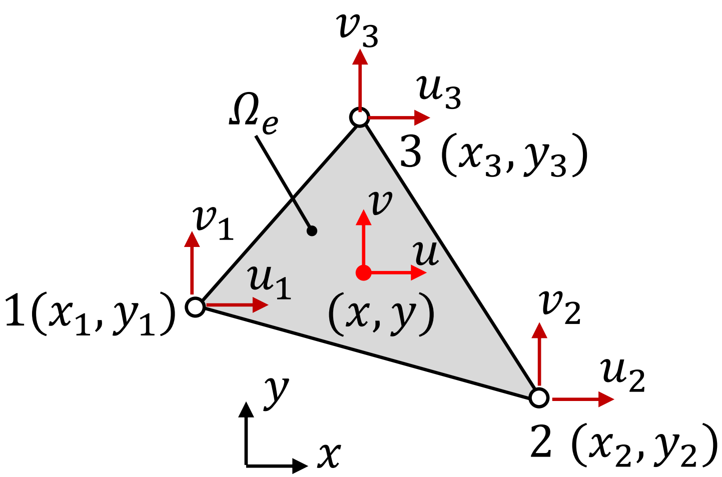
Model of a rectangular plate





one-quarter model with two finite elements

CST element in Plain Stress



shape functions = normalized area coordinates:

$$N_1(x,y) = \frac{A_1(x,y)}{A_e} = \frac{1}{2A_e}(a_1 + b_1 x + c_1 y)$$

$$N_2(x,y) = \frac{A_2(x,y)}{A_e} = \frac{1}{2A_e}(a_2 + b_2 x + c_2 y)$$

$$N_3(x,y) = \frac{A_3(x,y)}{A_e} = \frac{1}{2A_e}(a_3 + b_3 x + c_3 y)$$

where:

$a_1 = x_2 y_3 - x_3 y_2$; $a_2 = x_3 y_1 - x_1 y_3$; $a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_2 - y_3$; $b_2 = y_3 - y_1$; $b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$; $c_2 = x_1 - x_3$; $c_3 = x_2 - x_1$

$a_i = x_j y_k - x_k y_j$

$b_i = y_j - y_k$

$c_i = x_k - x_j$

Strain−displacement matrix

$$[B] = \frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

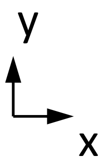
local stiffness matrix of the element:

$$\underset{6\times 6}{[k]_e} = A_e t_e \underset{6\times 3}{[B]^T}\underset{3\times 3}{[D]}\underset{3\times 6}{[B]}$$

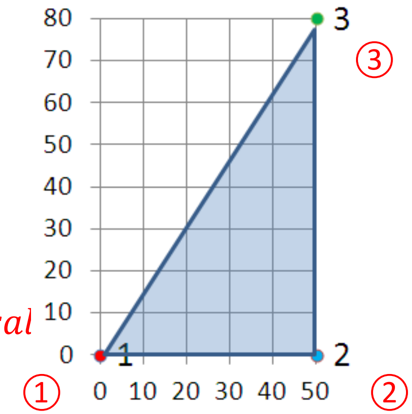
$$\underset{3\times 3}{[D]} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

Constitutive matrix for Plain stress

Calculating the element 1 matrices



notationlocal



Element	1	
E=	7.00E+04	MPa
ni=	0.33333333	
he=	2	mm
Ae=	2000	mm²

$$[B] = \frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$a_i = x_j y_k - x_k y_j$$
$$b_i = y_j - y_k$$
$$c_i = x_k - x_j$$

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

	node	x i	y i	x j	y j	x k	y k	ai	bi	ci
①	1	0	0	50	0	50	80	4000	-80	0
②	2	50	0	50	80	0	0	0	80	-50
③	3	50	80	0	0	50	0	0	0	50

$B_1=$

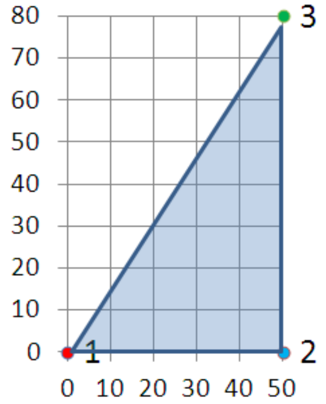
-0.02	0	0.02	0	0	0
0	0	0	-0.0125	0	0.0125
0	-0.02	-0.0125	0.02	0.0125	0

$B_1^T=$

-0.02	0	0
0	0	-0.02
0.02	0	-0.0125
0	-0.0125	0.02
0	0	0.0125
0	0.0125	0

$D=$

78750	26250	0
26250	78750	0
0	0	26250



Calculating the element 1 matrices

$$B_1^T D B_1 =$$

$6 \times 3 \quad 3 \times 3 \quad 3 \times 6$

31.5	0	-31.5	6.5625	0	-6.5625
0	10.5	6.5625	-10.5	-6.5625	0
-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
0	-6.5625	-4.10156	6.5625	4.101563	0
-6.5625	0	6.5625	-12.3047	0	12.30469

Matrix multiplication example:

D

78750	26250	0
26250	78750	0
0	0	26250

B_1

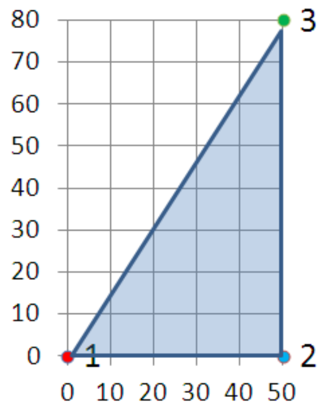
-0.02	0	0.02	0	0	0
0	0	0	-0.0125	0	0.0125
0	-0.02	-0.0125	0.02	0.0125	0

B_1^T

-0.02	0	0
0	0	-0.02
0.02	0	-0.0125
0	-0.0125	0.02
0	0	0.0125
0	0.0125	0

31.5	0	-31.5	6.5625	0	-6.5625
0	10.5	6.5625	-10.5	-6.5625	0
-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
0	-6.5625	-4.10156	6.5625	4.101563	0
-6.5625	0	6.5625	-12.3047	0	12.30469

Calculation of the stiffness matrix of element 1



$B_1{}^T D B_1 =$

31.5	0	-31.5	6.5625	0	-6.5625
0	10.5	6.5625	-10.5	-6.5625	0
-31.5	6.5625	35.60156	-13.125	-4.10156	6.5625
6.5625	-10.5	-13.125	22.80469	6.5625	-12.3047
0	-6.5625	-4.10156	6.5625	4.101563	0
-6.5625	0	6.5625	-12.3047	0	12.30469

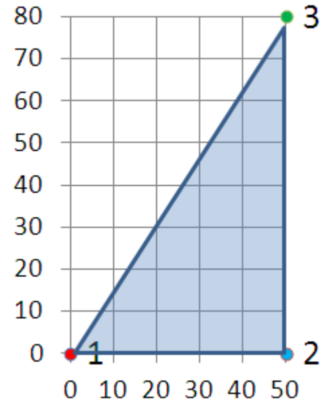
$$\underset{6\times 6}{[k]_e} = A_e t_e \underset{6\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 6}{[B]}$$

element matrix of element 1:

$k_1 =$

		u	v	u	v	u	v
		1	**1**	**2**	**2**	**3**	**3**
u	**1**	126000	0	-126000	26250	0	-26250
v	**1**	0	42000	26250	-42000	-26250	0
u	**2**	-126000	26250	142406.3	-52500	-16406.3	26250
v	**2**	26250	-42000	-52500	91218.75	26250	-49218.8
u	**3**	0	-26250	-16406.3	26250	16406.25	0
v	**3**	-26250	0	26250	-49218.8	0	49218.75

Determination of the extended stiffness matrix of element 1

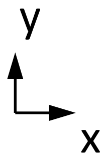


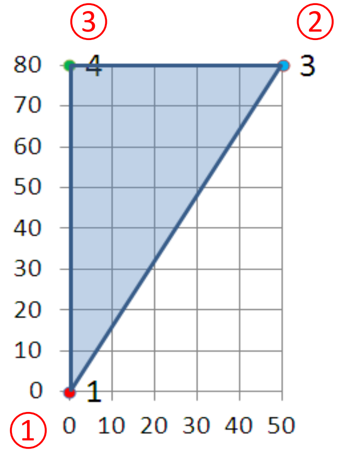
extended stiffness matrix of element 1:

$k_1^*=$

	u1	v1	u2	v2	u3	v3	u4	v4
u1	126000	0	-126000	26250	0	-26250	0	0
v1	0	42000	26250	-42000	-26250	0	0	0
u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
u3	0	-26250	-16406.3	26250	16406.25	0	0	0
v3	-26250	0	26250	-49218.8	0	49218.75	0	0
u4	0	0	0	0	0	0	0	0
v4	0	0	0	0	0	0	0	0

Calculating the element 2 matrices





local notation

Element	2	
E=	7.00E+04	MPa
ni=	0.33333333	
he=	2	mm
Ae=	2000	mm^2
p_{max}=	60	MPa

$$a_i = x_j y_k - x_k y_j$$
$$b_i = y_j - y_k$$
$$c_i = x_k - x_j$$

$$[B] = \frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

	node	x i	y i	x j	y j	x k	y k	ai	bi	ci
①	1	0	0	50	80	0	80	4000	0	-50
②	3	50	80	0	80	0	0	0	80	0
③	4	0	80	0	0	50	80	0	-80	50

B_2=

0	0	0.02	0	-0.02	0
0	-0.0125	0	0	0	0.0125
-0.0125	0	0	0.02	0.0125	-0.02

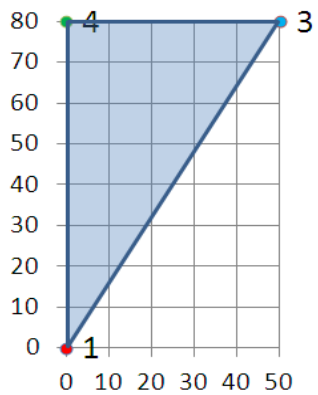
B_2^T=

0	0	-0.0125
0	-0.0125	0
0.02	0	0
0	0	0.02
-0.02	0	0.0125
0	0.0125	-0.02

D=

78750	26250	0
26250	78750	0
0	0	26250

Calculation of the stiffness matrix of element 2



$B_2^T D B_2 =$

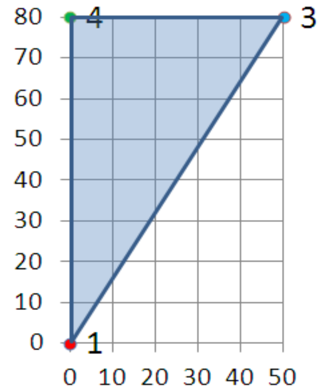
4.1015625	0	0	-6.5625	-4.10156	6.5625
0	12.30469	-6.5625	0	6.5625	-12.3047
0	-6.5625	31.5	0	-31.5	6.5625
-6.5625	0	0	10.5	6.5625	-10.5
-4.1015625	6.5625	-31.5	6.5625	35.60156	-13.125
6.5625	-12.3047	6.5625	-10.5	-13.125	22.80469

element matrix of element 2:

$k_2 =$

		u	v	u	v	u	v
		1	1	3	3	4	4
u	1	16406.25	0	0	-26250	-16406.3	26250
v	1	0	49218.75	-26250	0	26250	-49218.8
u	3	0	-26250	126000	0	-126000	26250
v	3	-26250	0	0	42000	26250	-42000
u	4	-16406.25	26250	-126000	26250	142406.3	-52500
v	4	26250	-49218.75	26250	-42000	-52500	91218.75

Determination of the extended element stiffness matrix 2



extended stiffness matrix of element 2:

$k_2^*=$

	u1	v1	u2	v2	u3	v3	u4	v4
u1	16406.25	0	0	0	0	-26250	-16406.3	26250
v1	0	49218.75	0	0	-26250	0	26250	-49218.75
u2	0	0	0	0	0	0	0	0
v2	0	0	0	0	0	0	0	0
u3	0	-26250	0	0	126000	0	-126000	26250
v3	-26250	0	0	0	0	42000	26250	-42000
u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

Determination of the global stiffness matrix

extended stiffness matrix of element 1:

$k_1^*=$

	u1	v1	u2	v2	u3	v3	u4	v4
u1	126000	0	-126000	26250	0	-26250	0	0
v1	0	42000	26250	-42000	-26250	0	0	0
u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
u3	0	-26250	-16406.3	26250	16406.25	0	0	0
v3	-26250	0	26250	-49218.8	0	49218.75	0	0
u4	0	0	0	0	0	0	0	0
v4	0	0	0	0	0	0	0	0

extended stiffness matrix of element 2:

$k_2^*=$

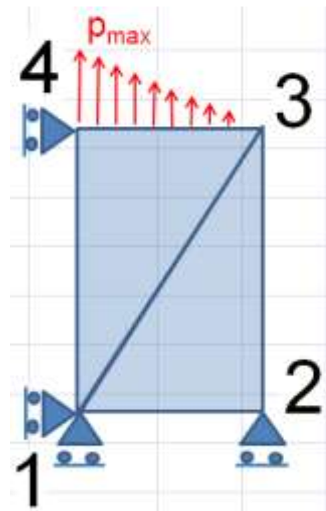
	u1	v1	u2	v2	u3	v3	u4	v4
u1	16406.25	0	0	0	0	-26250	-16406.3	26250
v1	0	49218.75	0	0	-26250	0	26250	-49218.75
u2	0	0	0	0	0	0	0	0
v2	0	0	0	0	0	0	0	0
u3	0	-26250	0	0	126000	0	-126000	26250
v3	-26250	0	0	0	0	42000	26250	-42000
u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

global stiffness matrix:

$K=$ (NDOF x NDOF)

	u1	v1	u2	v2	u3	v3	u4	v4
u1	142406.3	0	-126000	26250	0	-52500	-16406.3	26250
v1	0	91218.75	26250	-42000	-52500	0	26250	-49218.75
u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0
v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0
u3	0	-52500	-16406.3	26250	142406.3	0	-126000	26250
v3	-52500	0	26250	-49218.8	0	91218.75	26250	-42000
u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500
v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75

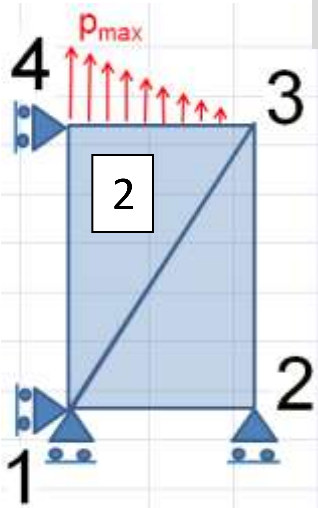
Introduction of boundary conditions to the global stiffness matrix



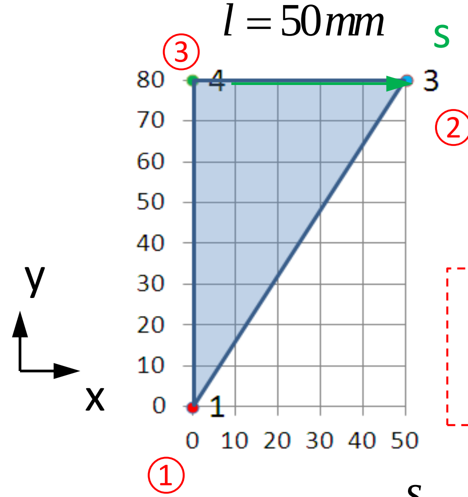
K= (NDOF x NDOF)

	u1	v1	u2	v2	u3	v3	u4	v4	Boundary conditions
u1	142406.3	0	-126000	26250	0	-52500	-16406.3	26250	u1 =0
v1	0	91218.75	26250	-42000	-52500	0	26250	-49218.75	v1 =0
u2	-126000	26250	142406.3	-52500	-16406.3	26250	0	0	
v2	26250	-42000	-52500	91218.75	26250	-49218.8	0	0	v2 =0
u3	0	-52500	-16406.3	26250	142406.3	0	-126000	26250	
v3	-52500	0	26250	-49218.8	0	91218.75	26250	-42000	
u4	-16406.3	26250	0	0	-126000	26250	142406.3	-52500	u4 =0
v4	26250	-49218.8	0	0	26250	-42000	-52500	91218.75	

Equivalent load vector of surface loads



$l = 50mm$

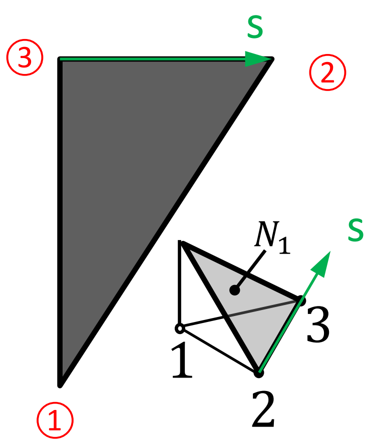


$$\lfloor p \rfloor = \lfloor p_x, \quad p_y \rfloor = \lfloor 0, \quad p(s) \rfloor = \left\lfloor 0, \quad p_{max}(1 - \frac{s}{l}) \right\rfloor$$

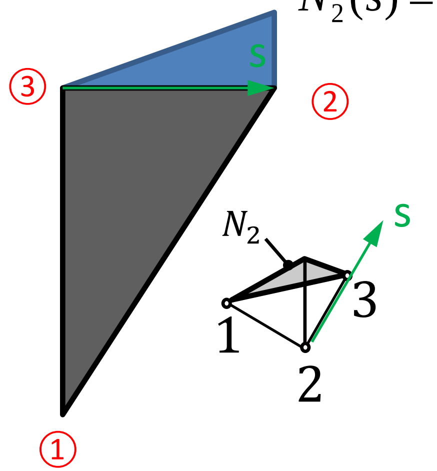
p_{max}=60MPa

$$\lfloor F^p \rfloor_e = t_e \int_0^l \lfloor p_x, p_y \rfloor \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} ds$$

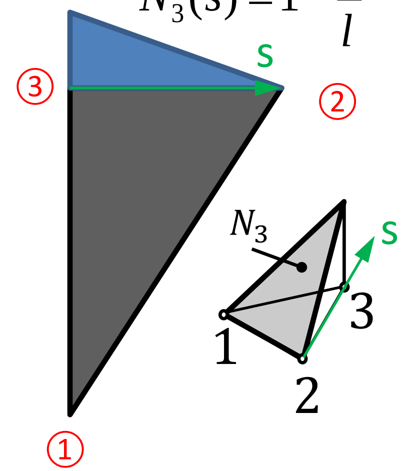
$N_1(s) = 0$



$N_2(s) = \frac{s}{l}$



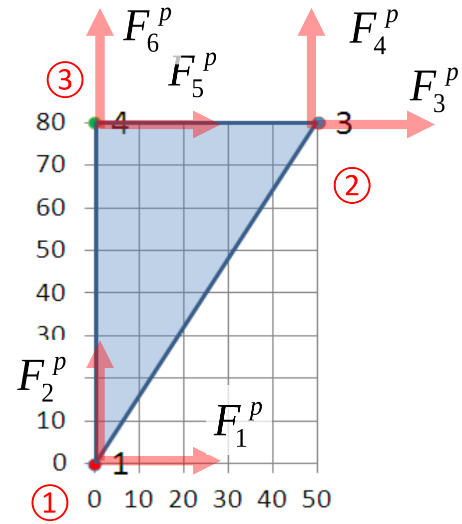
$N_3(s) = 1 - \frac{s}{l}$



$$\lfloor F^p \rfloor_2 = h \int_0^l \lfloor p_x(s), \quad p_y(s) \rfloor \begin{bmatrix} N_1(s) & 0 & N_2(s) & 0 & N_3(s) & 0 \\ 0 & N_1(s) & 0 & N_2(s) & 0 & N_3(s) \end{bmatrix} ds =$$

$$= \lfloor F_1^{\,p}, \quad F_2^{\,p}, \quad F_3^{\,p}, \quad F_4^{\,p}, \quad F_5^{\,p}, \quad F_6^{\,p} \rfloor_2$$

Equivalent load vector of surface loads



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$$F_1^{\,p} = h\int_0^l p_x(s)N_1(s)\,ds = h\int_0^l 0\cdot 0\,ds = 0$$

$$F_2^{\,p} = h\int_0^l p_y(s)N_1(s)\,ds = h\int_0^l p_{\max}(1-\frac{s}{l})\cdot 0\,ds = 0$$

$$F_3^{\,p} = h\int_0^l p_x(s)N_2(s)\,ds = h\int_0^l 0\cdot\frac{s}{l_1}\,ds = 0$$

$$F_4^{\,p} = h\int_0^l p_y(s)N_2(s)\,ds = h\int_0^l p_{\max}(1-\frac{s}{l})\cdot\frac{s}{l}\,ds = \frac{1}{6}p_{\max}lh = 1000\,N$$

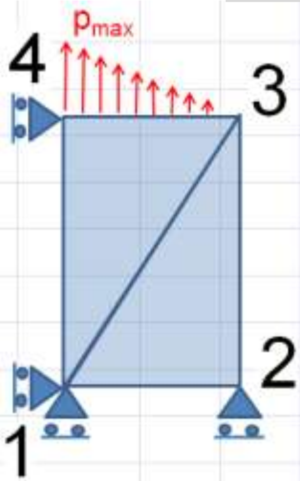
$$F_5^{\,p} = h\int_0^l p_x(s)N_3(s)\,ds = h\int_0^l 0\cdot(1-\frac{s}{l})\,ds = 0$$

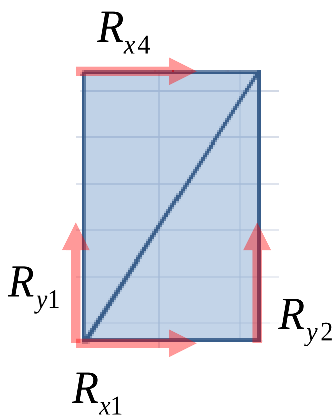
$$F_6^{\,p} = h\int_0^l p_y(s)N_3(s)\,ds = h\int_0^l p_{\max}(1-\frac{s}{l})\cdot(1-\frac{s}{l})\,ds = \frac{1}{3}p_{\max}lh = 2000\,N$$

F^e =

0
0
0
0
0
1000
0
2000

Determination of nodal displacements





$F^e=$

0
0
0
0
0
1000
0
2000

$F^n=$

Rx1
Ry1
0
Ry2
0
0
Rx4
0

$\underset{NDOF \times 1}{F}=$

Rx1
Ry1
0
Ry2
0
1000
Rx4
2000

$\underset{N \times N}{K}=$

	u2	u3	v3	v4
u2	142406.3	-16406.3	26250	0
u3	-16406.3	142406.3	0	26250
v3	26250	0	91218.75	-42000
v4	0	26250	-42000	91218.75

$\underset{N \times N}{K^{-1}}=$

	u2	u3	v3	v4
u2	7.71864E-06	1.20993E-06	-3.02221E-06	-1.7397E-06
u3	1.20993E-06	7.71864E-06	-1.7397E-06	-3.02221E-06
v3	-3.02221E-06	-1.7397E-06	1.53082E-05	7.54899E-06
v4	-1.7397E-06	-3.02221E-06	7.54899E-06	1.53082E-05

$\underset{N \times 1}{F}=$

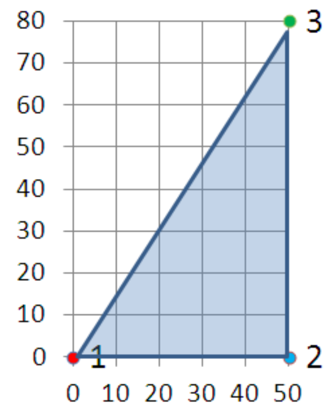
0
0
1000
2000

$K^{-1}F$ ⇒

$\underset{N \times 1}{q}=$

-0.006502	mm	u2
-0.007784	mm	u3
0.030406	mm	v3
0.038165	mm	v4

Determination of strain and stress in element 1



$B_1 =$

-0.02	0	0.02	0	0	0
0	0	0	-0.0125	0	0.0125
0	-0.02	-0.0125	0.02	0.0125	0

$q_1 =$ (ne x 1)

0	mm	u1
0	mm	v1
-0.006502	mm	u2
0	mm	v2
-0.007784	mm	u3
0.030406	mm	v3

$B_1 q_1$

$D =$

78750	26250	0
26250	78750	0
0	0	26250

$\varepsilon_1 =$

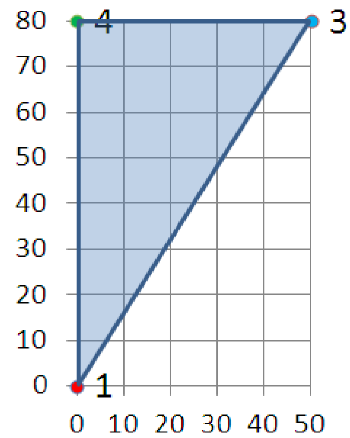
-0.000130032
0.000380077
-1.60313E-05

$D\varepsilon_1$ →

$\sigma_1 =$

-0.263	MPa
26.52	MPa
-0.421	MPa

Determination of strain and stress in element 2



$B_2=$

0	0	0.02	0	-0.02	0
0	-0.0125	0	0	0	0.0125
-0.0125	0	0	0.02	0.0125	-0.02

$q_2=$ (ne x 1)

0	mm	u1
0	mm	v1
-0.007784	mm	u3
0.030406	mm	v3
0	mm	u4
0.038165	mm	v4

$B_2 q_2$

$D=$

78750	26250	0
26250	78750	0
0	0	26250

$\varepsilon_2=$

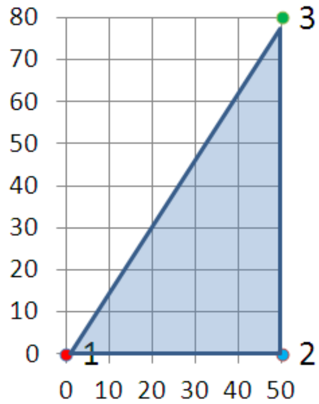
-0.000155682
0.000477066
-0.000155183

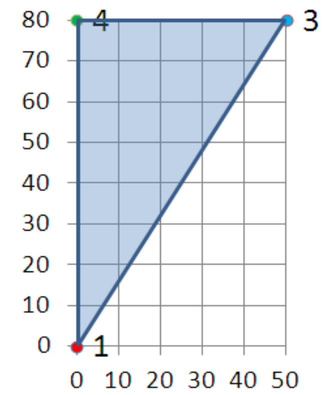
$D\varepsilon_2$

$\sigma_2=$

0.263	MPa
33.48	MPa
-4.074	MPa

Determination of elastic strain energy in elements





$$U_e = \frac{1}{2} \int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}} \, d\Omega_e = \frac{1}{2} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}} \int_{\Omega_e} d\Omega_e$$

$U_1 =$ 20.23940803 Nmm

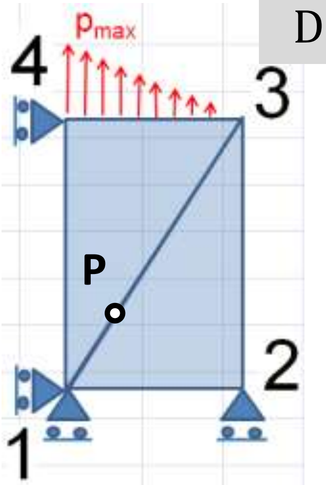
$U_2 =$ 33.12895375 Nmm

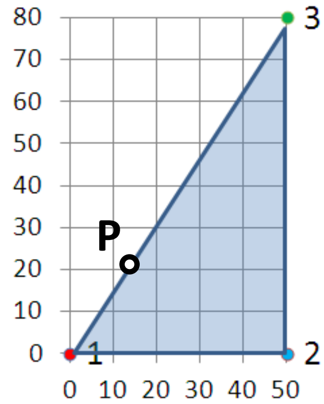
$U =$ 53.37 Nmm

$U_{exact} =$ 59.35 Nmm

$U =$ 89.93% U_{exact}

Determination of solutions at point P on the boundary of elements





Element 1

node	x i	y i	x j	y j	x k	y k	ai	bi	ci
1	0	0	50	0	50	80	4000	-80	0
2	50	0	50	80	0	0	0	80	-50
3	50	80	0	0	50	0	0	0	50

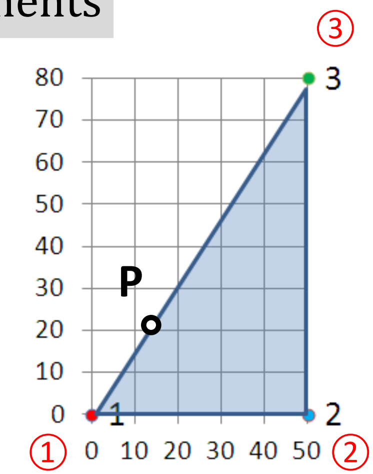
$$N_1(x_P, y_P) = N_1(12.5, 20) = \frac{a_1 + b_1 x_P + c_1 y_P}{2 \cdot A_e} = \frac{4000\,mm^2 + (-80\,mm) \cdot 12.5\,mm + 0\,mm \cdot 20\,mm}{2 \cdot 2000\,mm^2} = \frac{3}{4}$$

$$N_2(x_P, y_P) = N_2(12.5, 20) = \frac{a_2 + b_2 x_P + c_2 y_P}{2 \cdot A_e} = \frac{0 + 80 \cdot 12.5 + (-50) \cdot 20}{2 \cdot 2000} = 0$$

$$N_3(x_P, y_P) = N_3(12.5, 20) = \frac{a_3 + b_3 x_P + c_3 y_P}{2 \cdot A_e} = \frac{0 + 0 \cdot 12.5 + 50 \cdot 20}{2 \cdot 2000} = \frac{1}{4}$$

Determination of solutions at point P on the boundary of elements

$$N_1(12.5,20)+N_2(12.5,20)+N_3(12.5,20)=\frac{3}{4}+0+\frac{1}{4}=1$$



$$x=\sum_{i=1}^{3}N_i(x,y)\cdot x_i \Rightarrow x_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot x_i = N_1\cdot x_1+N_2\cdot x_2+N_3\cdot x_3=$$

$$=\frac{3}{4}\cdot 0+0\cdot 50+\frac{1}{4}\cdot 50=12.5\ mm$$

$$y=\sum_{i=1}^{3}N_i(x,y)\cdot y_i \Rightarrow y_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot y_i = N_1\cdot y_1+N_2\cdot y_2+N_3\cdot y_3=$$

$$=\frac{3}{4}\cdot 0+0\cdot 0+\frac{1}{4}\cdot 80=20\ mm$$

$q_1=$ (ne × 1)

0	mm	u1
0	mm	v1
-0.006502	mm	u2
0	mm	v2
-0.007784	mm	u3
0.030406	mm	v3

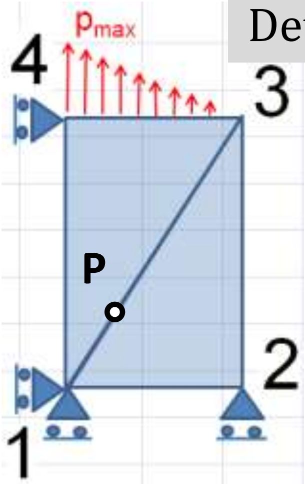
$$u=\sum_{i=1}^{3}N_i(x,y)\cdot u_i \Rightarrow u_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot u_i = N_1\cdot u_1+N_2\cdot u_2+N_3\cdot u_3=$$

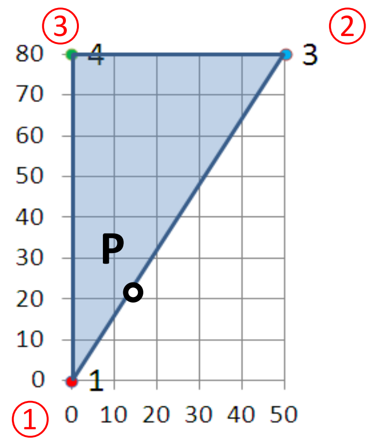
$$=\frac{3}{4}\cdot 0+0\cdot(-0.006502)+\frac{1}{4}\cdot(-0.007784)=-0.00195\ mm$$

$$v=\sum_{i=1}^{3}N_i(x,y)\cdot v_i \Rightarrow v_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot v_i = N_1\cdot v_1+N_2\cdot v_2+N_3\cdot v_3=$$

$$=\frac{3}{4}\cdot 0+0\cdot 0+\frac{1}{4}\cdot 0.030406=0.0076\ mm$$

Determination of solutions at point P on the boundary of elements





Element 2

node	x i	y i	x j	y j	x k	y k	ai	bi	ci
1	0	0	50	80	0	80	4000	0	-50
3	50	80	0	80	0	0	0	80	0
4	0	80	0	0	50	80	0	-80	50

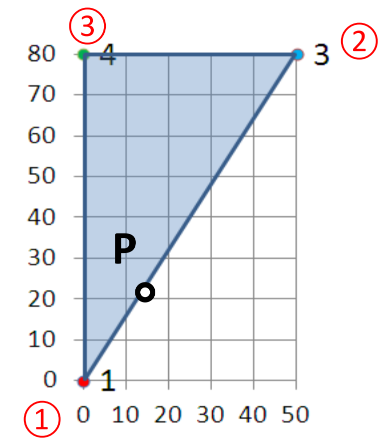
$$N_1(x_P, y_P) = N_1(12.5,20) = \frac{a_1 + b_1 x_P + c_1 y_P}{2 \cdot A_e} = \frac{4000\,mm^2 + 0mm \cdot 12.5mm + (-50\,mm) \cdot 20\,mm}{2 \cdot 2000\,mm^2} = \frac{3}{4}$$

$$N_2(x_P, y_P) = N_2(12.5,20) = \frac{a_2 + b_2 x_P + c_2 y_P}{2 \cdot A_e} = \frac{0 + 80 \cdot 12.5 + 0 \cdot 20}{2 \cdot 2000} = \frac{1}{4}$$

$$N_3(x_P, y_P) = N_3(12.5,20) = \frac{a_3 + b_3 x_P + c_3 y_P}{2 \cdot A_e} = \frac{0 + (-80) \cdot 12.5 + 50 \cdot 20}{2 \cdot 2000} = 0$$

Determination of solutions at point P on the boundary of elements

$$N_1(12.5,20)+N_2(12.5,20)+N_3(12.5,20)=\frac{3}{4}+\frac{1}{4}+0=1$$



$$x=\sum_{i=1}^{3}N_i(x,y)\cdot x_i \Rightarrow x_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot x_i=N_1\cdot x_1+N_2\cdot x_2+N_3\cdot x_3=$$

$$=\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 50+0\cdot 0=12.5\ mm$$

$$y=\sum_{i=1}^{3}N_i(x,y)\cdot y_i \Rightarrow y_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot y_i=N_1\cdot y_1+N_2\cdot y_2+N_3\cdot y_3=$$

$$=\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 80+0\cdot 80=20\ mm$$

local notation

$$u=\sum_{i=1}^{3}N_i(x,y)\cdot u_i \Rightarrow u_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot u_i=N_1\cdot u_1+N_2\cdot u_2+N_3\cdot u_3=$$

u_3 u_4 **global notation**

$$=\frac{3}{4}\cdot 0+\frac{1}{4}\cdot(-0.007784)+0\cdot 0=-0.00195\ mm$$

$$v=\sum_{i=1}^{3}N_i(x,y)\cdot v_i \Rightarrow v_p=\sum_{i=1}^{3}N_i(x_P,y_P)\cdot v_i=N_1\cdot v_1+N_2\cdot v_2+N_3\cdot v_3=$$

v_3 v_4 **global notation**

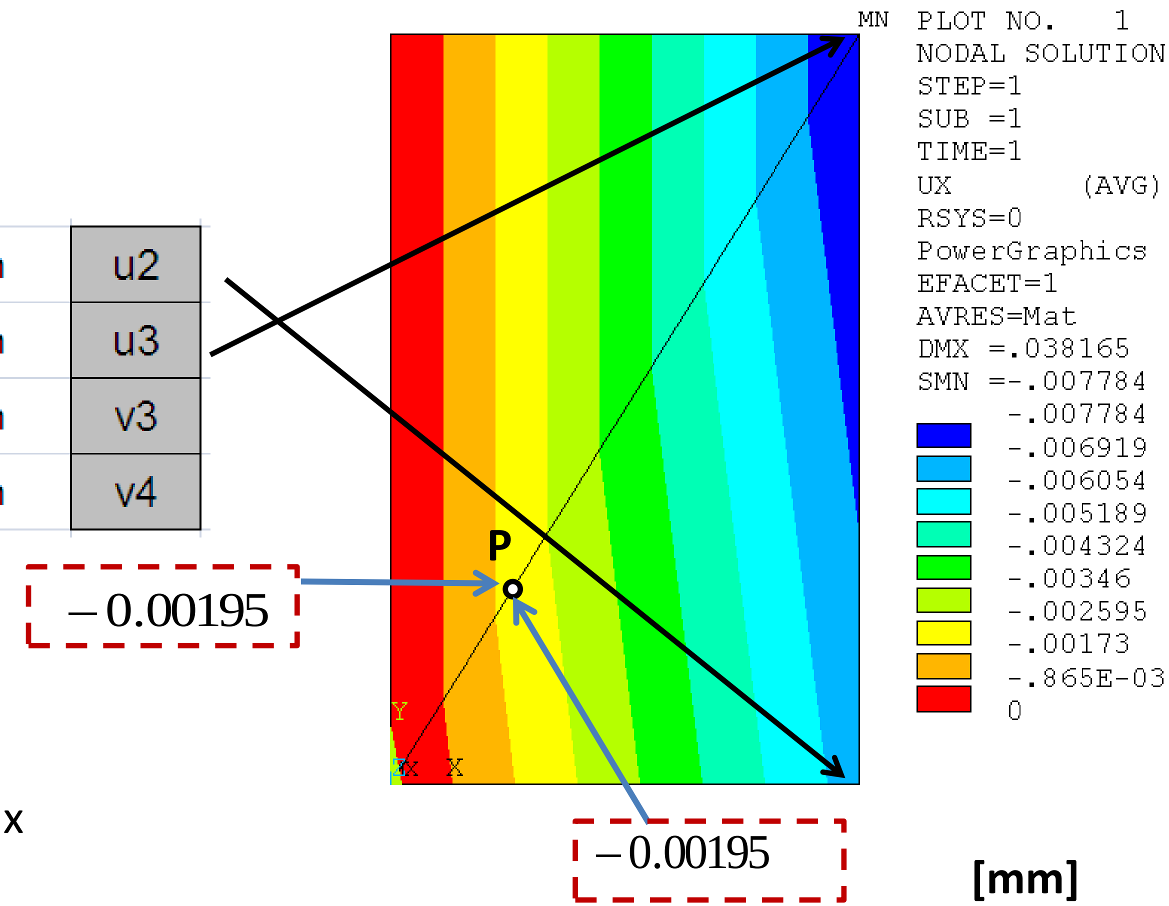
$$=\frac{3}{4}\cdot 0+\frac{1}{4}\cdot 0.030406+0\cdot 0.038165=0.0076\ mm$$

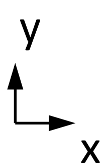
$\underset{ne\times 1}{q_2}=$

0	mm	u1
0	mm	v1
-0.007784	mm	u3
0.030406	mm	v3
0	mm	u4
0.038165	mm	v4

UX displacement

	-0.006502	mm	u2
$\underset{N\times 1}{q}=$	-0.007784	mm	u3
	0.030406	mm	v3
	0.038165	mm	v4



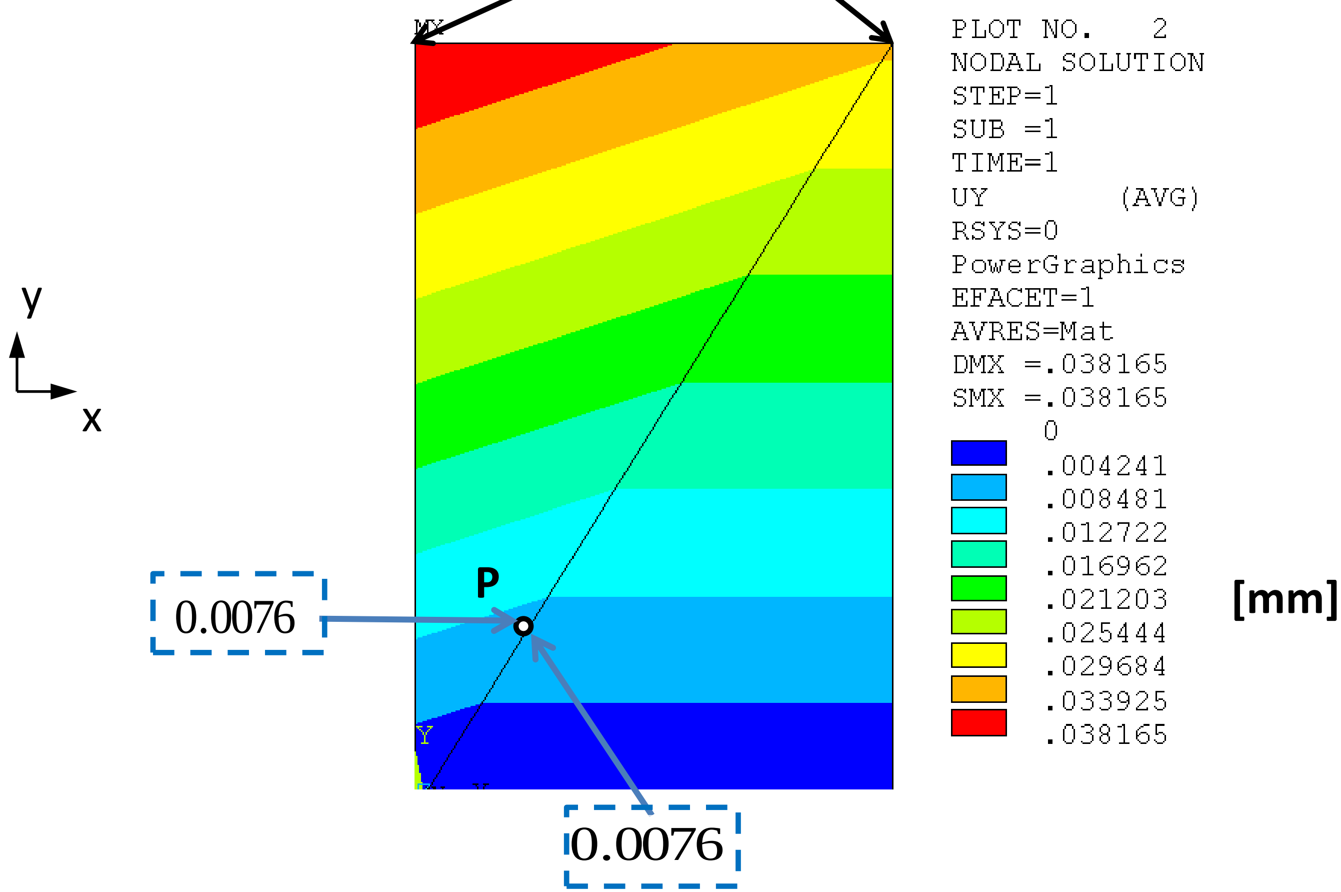


Displacements at point P on the boundary of elements

UY displacement

$\underset{N\times 1}{q} =$

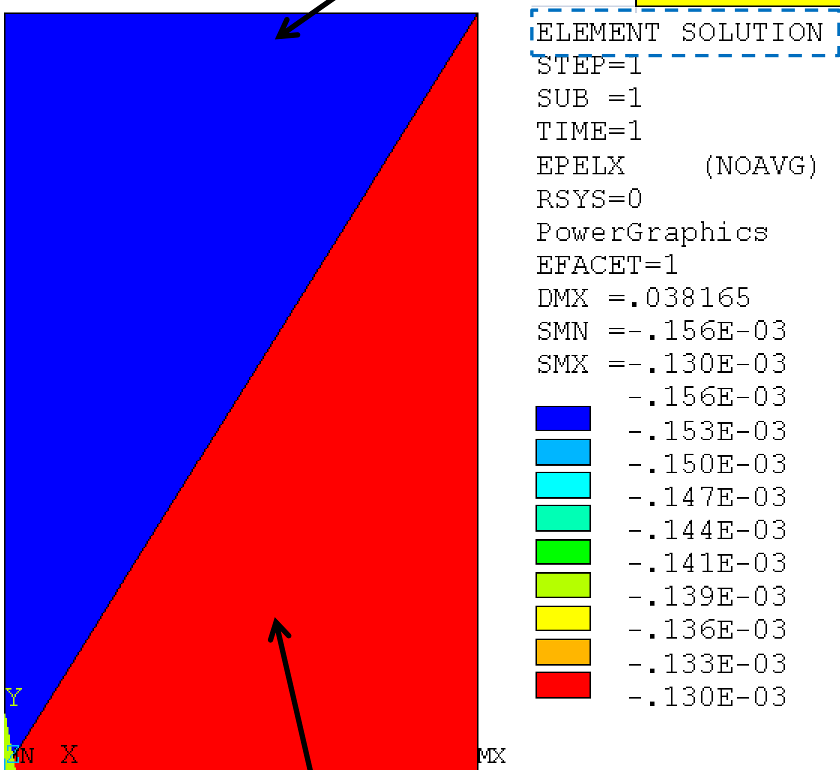
-0.006502	mm	u2
-0.007784	mm	u3
0.030406	mm	v3
0.038165	mm	v4



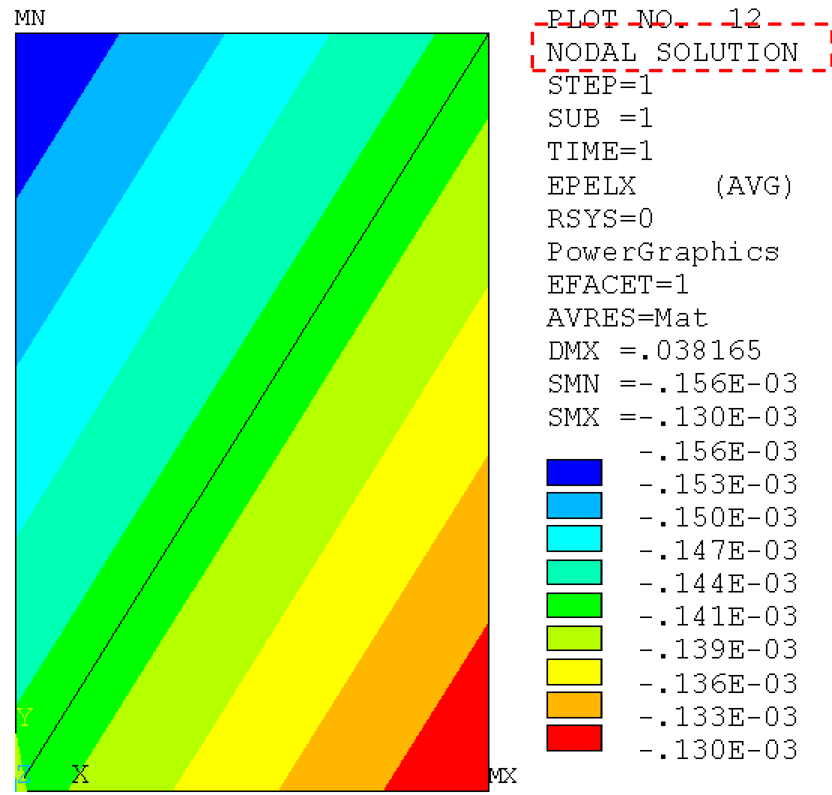
Strain in X at the boundary of elements

ε_x strain

	-0.000155682
$\varepsilon_2=$	0.000477066
	-0.000155183



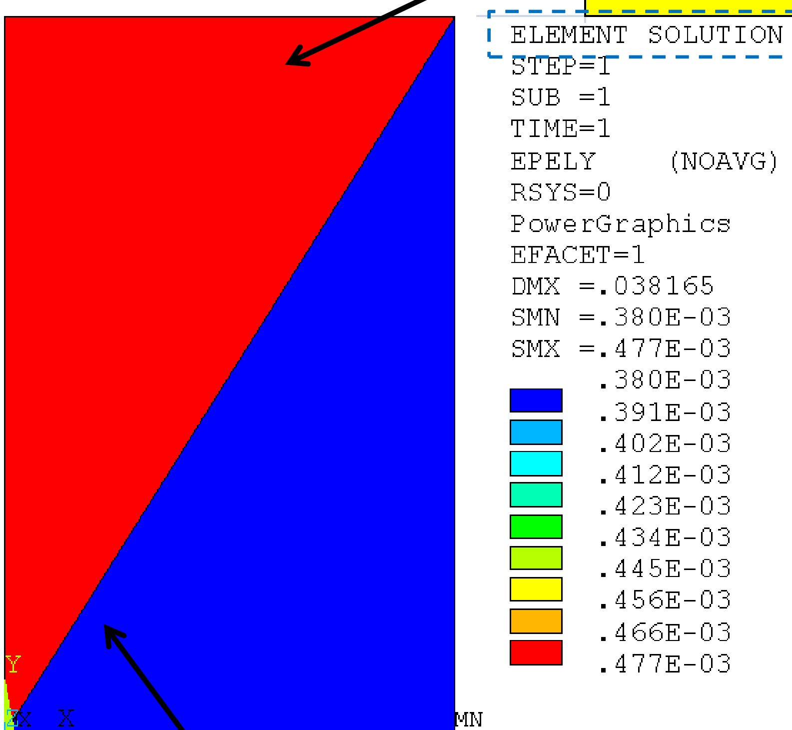
	-0.000130032
$\varepsilon_1=$	0.000380077
	-1.60313E-05

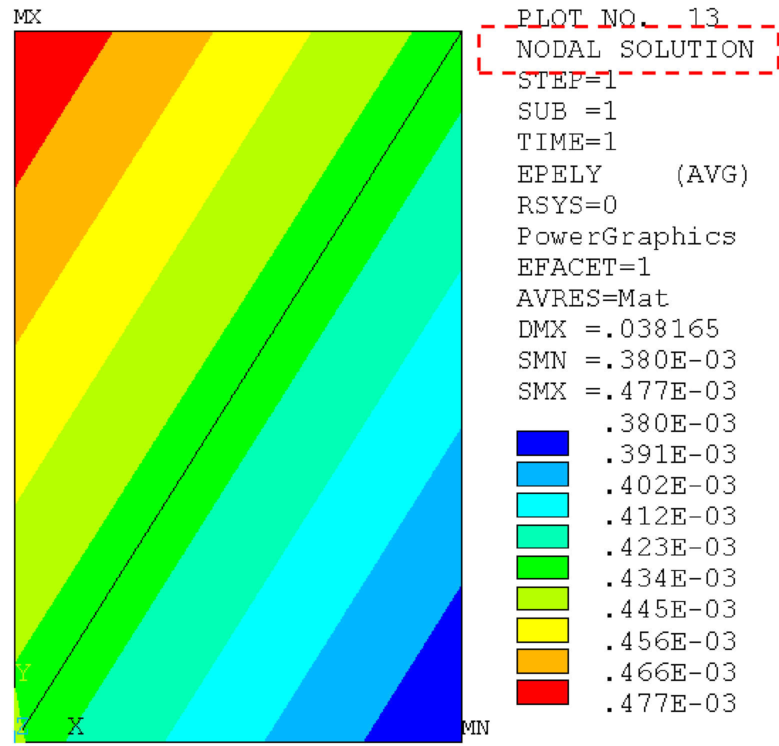


Strain in Y at the boundary of elements

ε_y **strain**

-0.000155682
ε_2= 0.000477066
-0.000155183

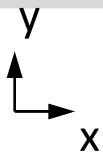




-0.000130032
ε_1= 0.000380077
-1.60313E-05

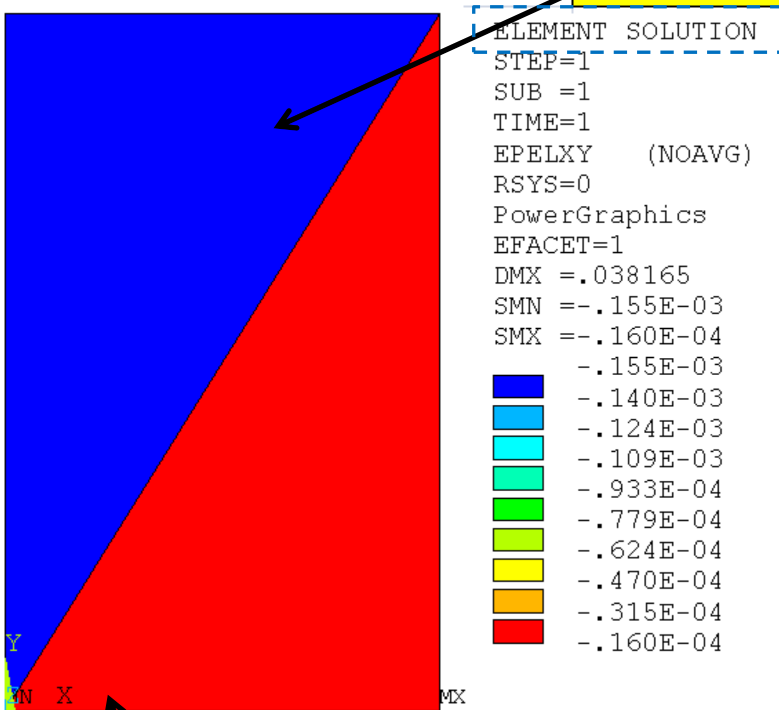
Shear strain at the boundary of elements

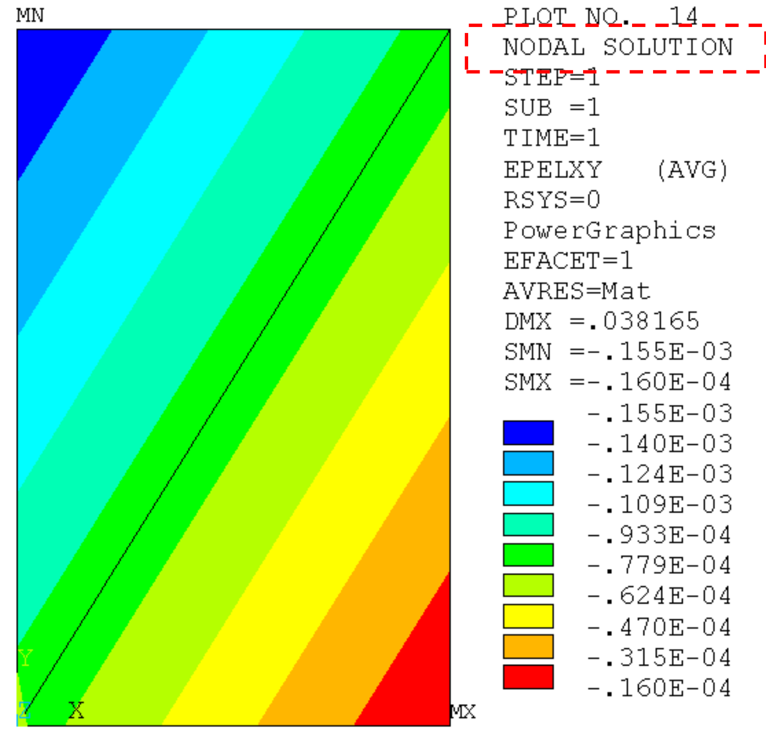
γ_{xy} **strain**



ε_2=

-0.000155682
0.000477066
-0.000155183



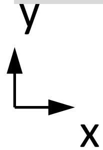


ε_1=

-0.000130032
0.000380077
-1.60313E-05

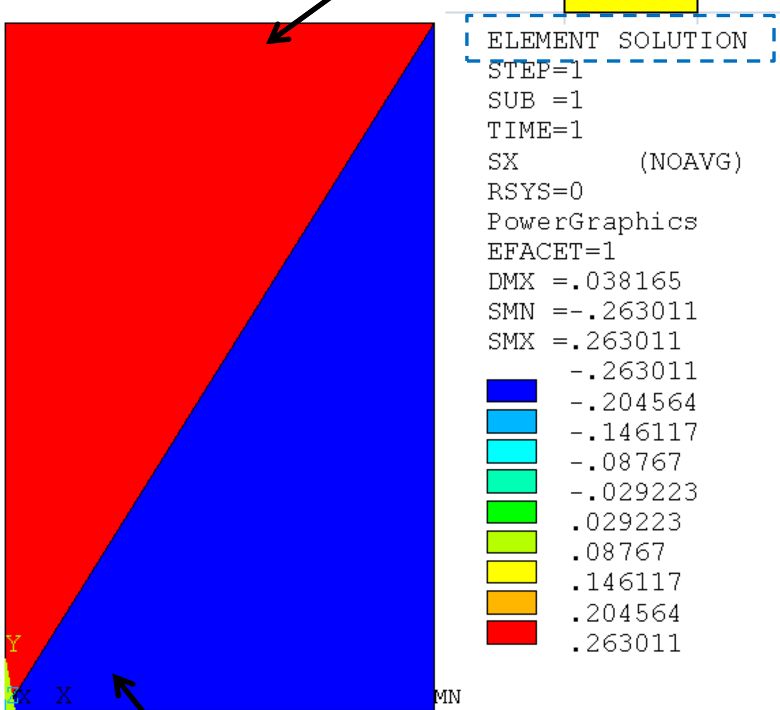
Stress in X at the boundary of elements

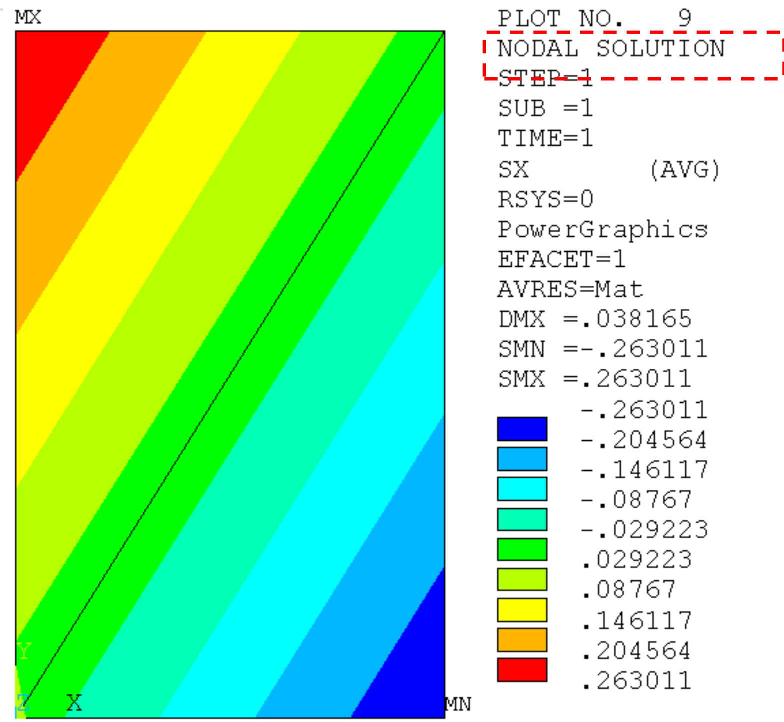
σ_x stress



$\sigma_2 =$

0.263	MPa
33.48	MPa
-4.074	MPa





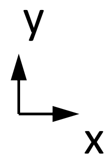
[MPa]

$\sigma_1 =$

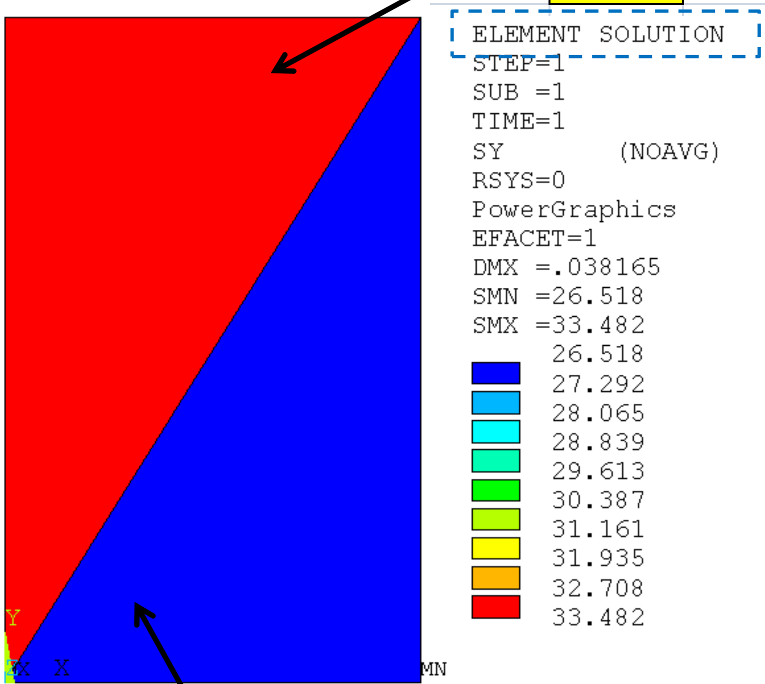
-0.263	MPa
26.52	MPa
-0.421	MPa

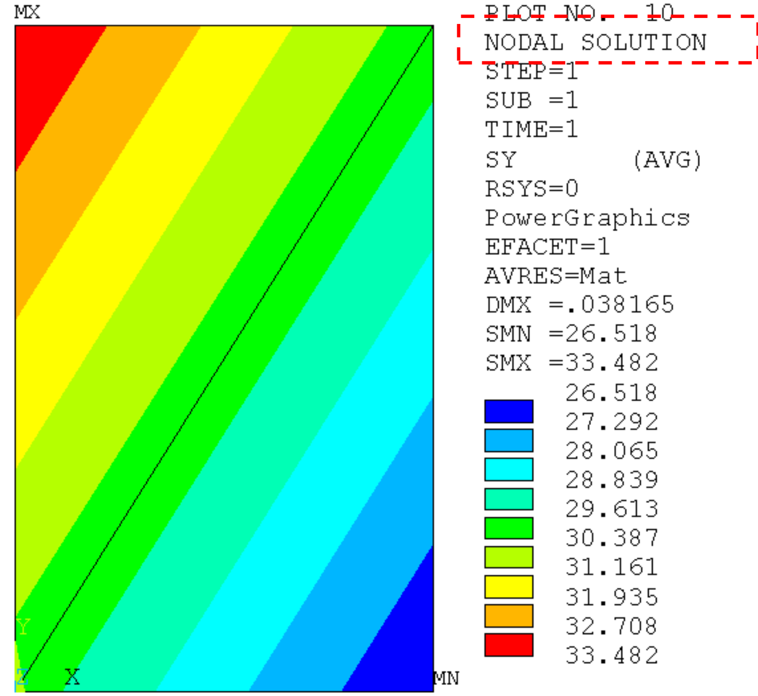
Stress in Y at the boundary of elements

σ_y stress



	0.263	MPa
σ_2=	33.48	MPa
	-4.074	MPa





	-0.263	MPa
σ_1=	26.52	MPa
	-0.421	MPa

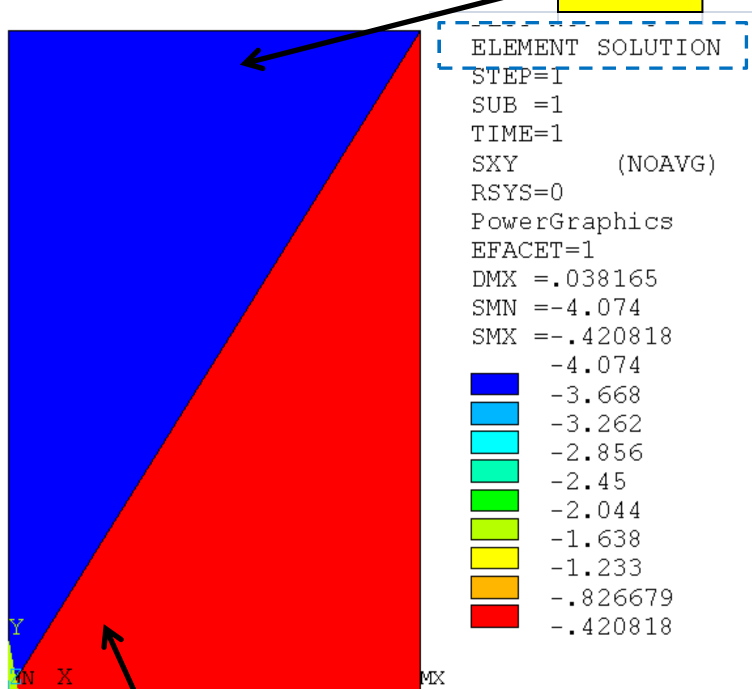
[MPa]

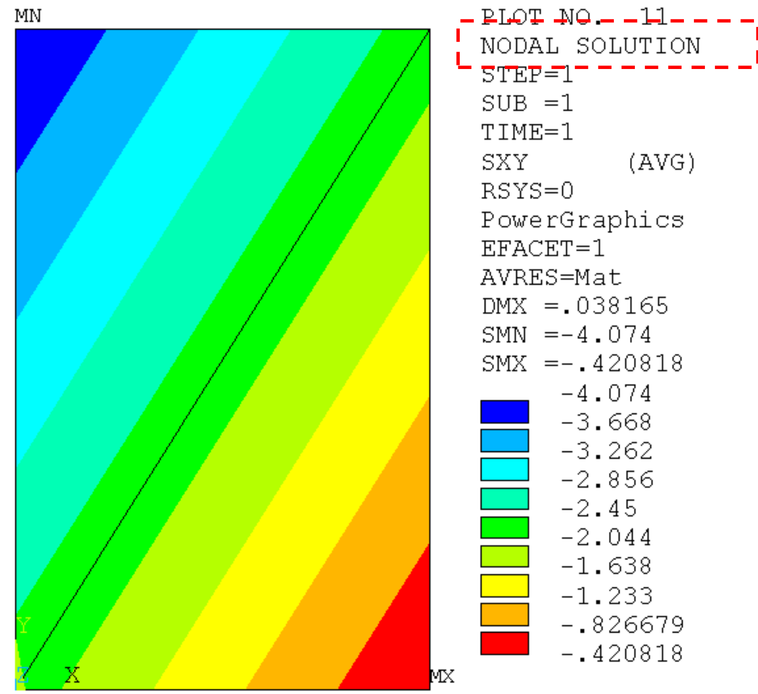
Shear stress at the boundary of elements

τ_{xy} stress (y, x)

$\sigma_2 =$

0.263	MPa
33.48	MPa
-4.074	MPa





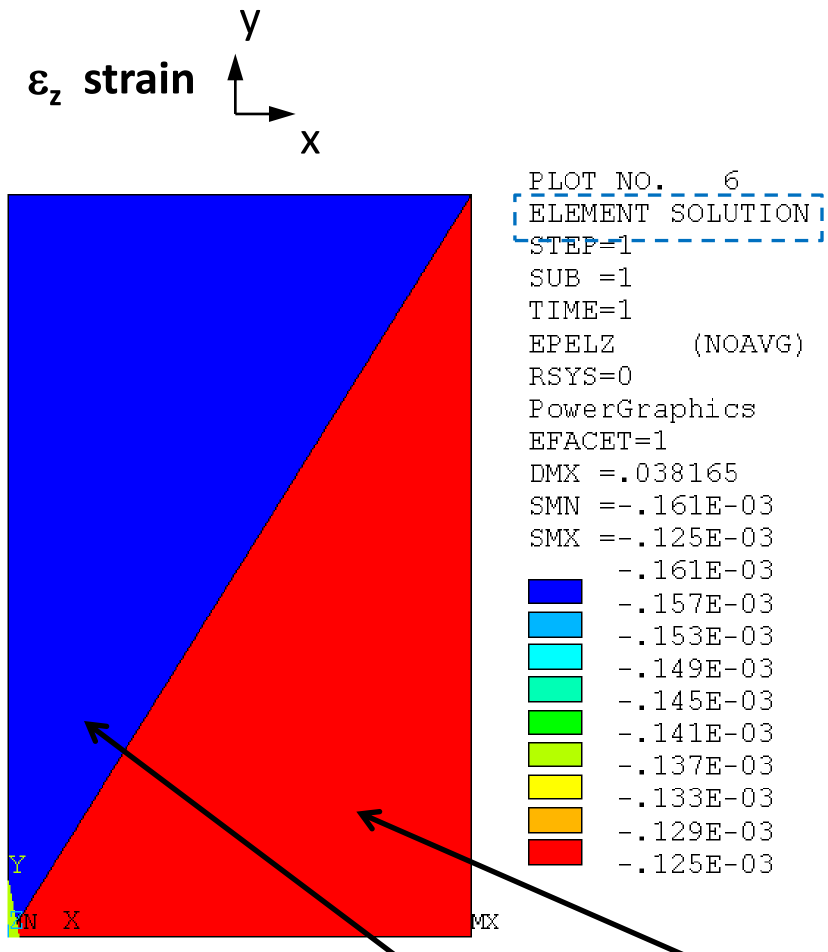
$\sigma_1 =$

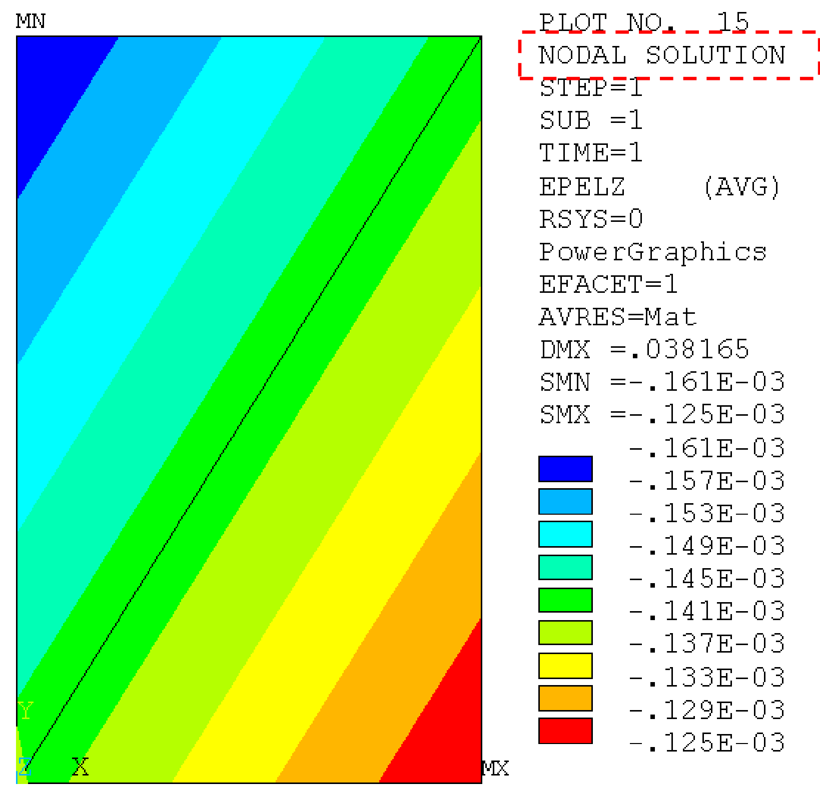
-0.263	MPa
26.52	MPa
-0.421	MPa

[MPa]

Strain in Z at the boundary of elements

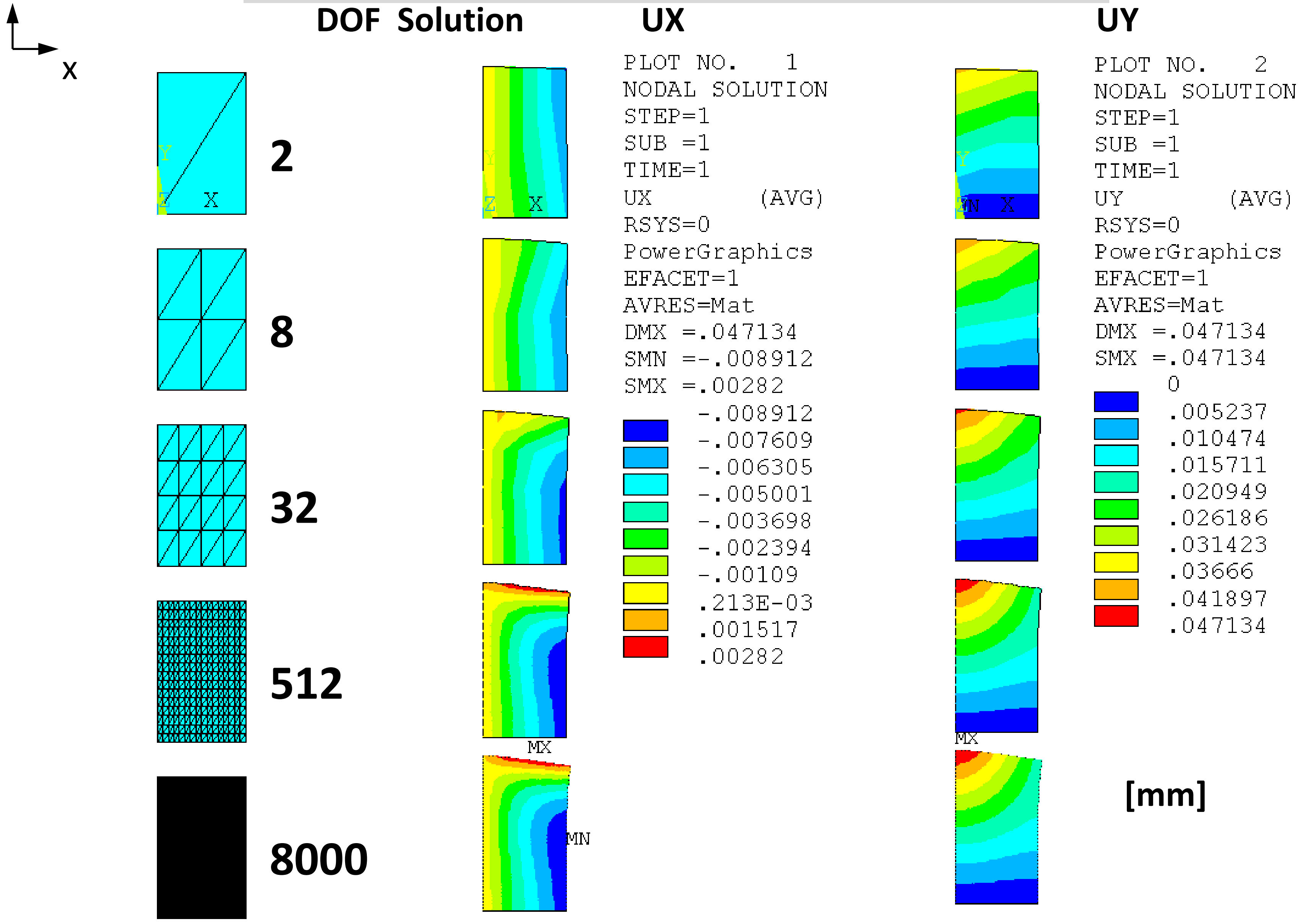
ε_z **strain**





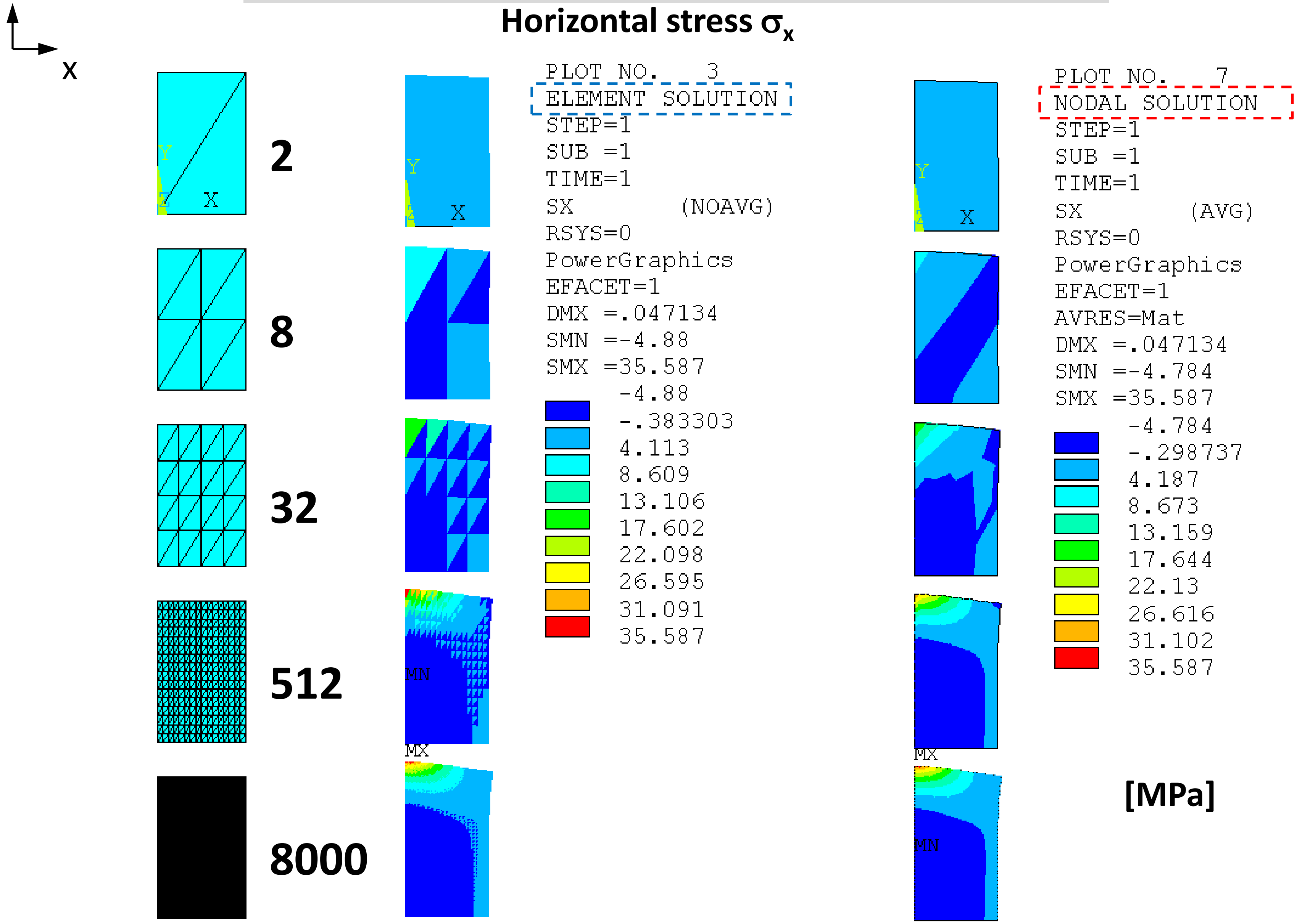
$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x+\sigma_y) = \begin{cases} -\dfrac{1}{3\cdot 7\cdot 10^4}(-0.263+26.52) = -0.125\cdot 10^{-3} \\ -\dfrac{1}{3\cdot 7\cdot 10^4}(0.263+33.48) = -0.161\cdot 10^{-3} \end{cases}$$

The impact of discretization on the quality of results



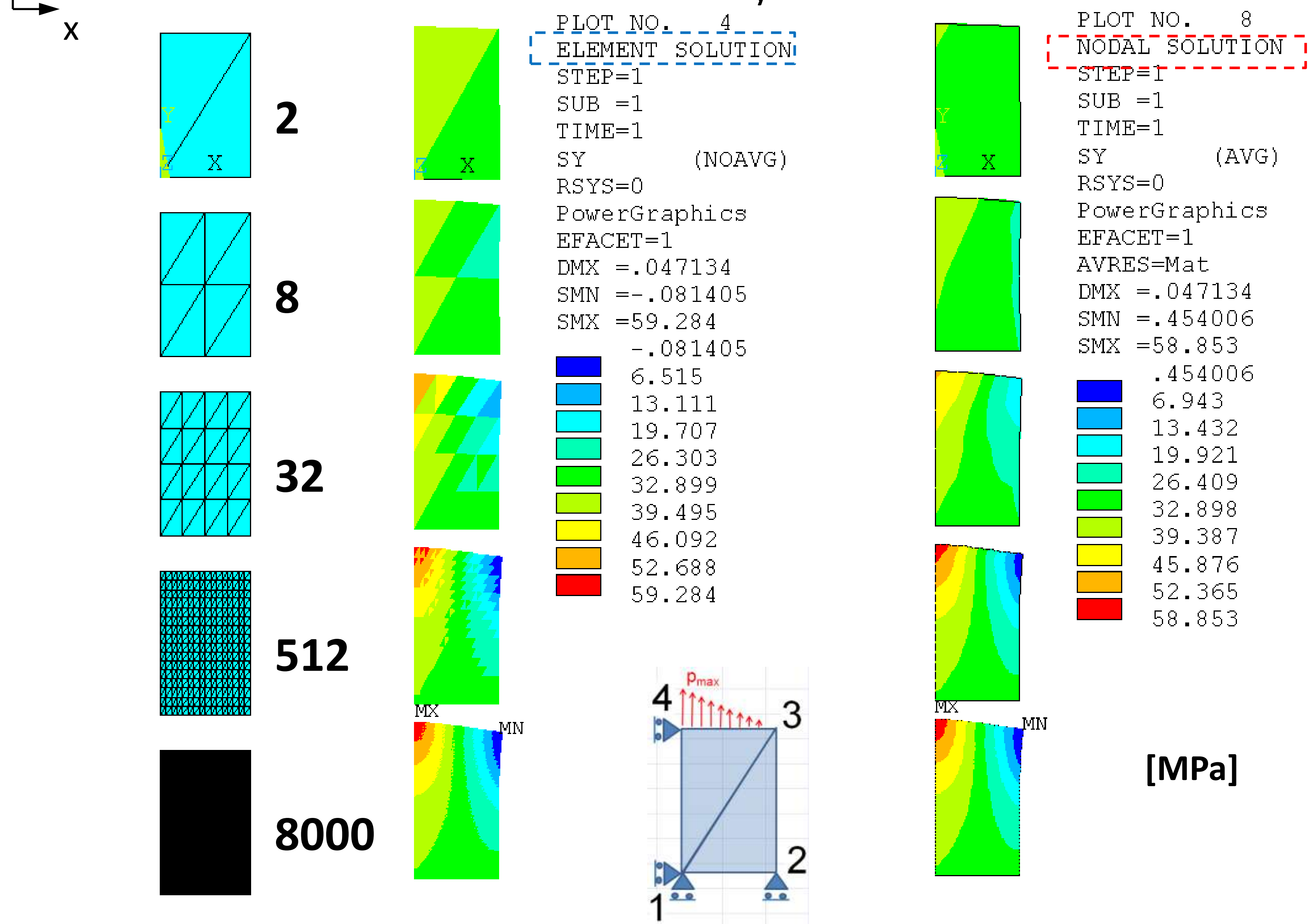
The impact of discretization on the quality of results

Horizontal stress σ_x



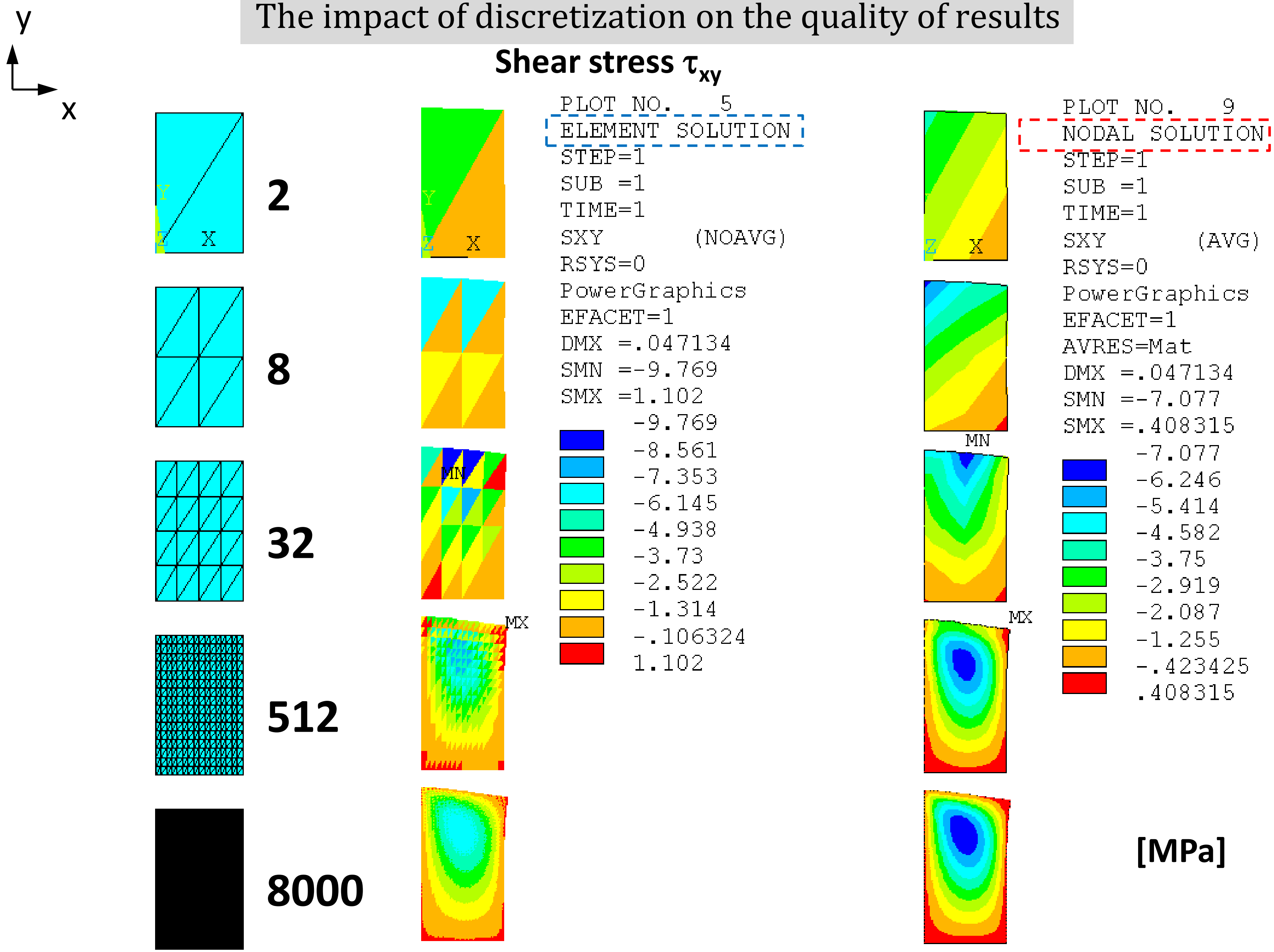
The impact of discretization on the quality of results

Vertical stress σ_y



The impact of discretization on the quality of results

Shear stress τ_{xy}



The impact of discretization on the quality of results